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Microphone and Loudspeaker Design

Lecture Notes 2022-2023

University of Salford

Contents

About these notes

These notes cover the key course content for the Level 5 Microphone and Loudspeaker Design module undertaken as part of the University of Salford's undergraduate degree in Acoustics and Audio Engineering.

Please be aware that these notes will be updated annually. If you have any comments, queries or corrections please email me at:
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Part I

Loudspeaker Design

List of symbols

Many of the symbols used in this module are shared between quantities in different domains (e.g. M can denote both mechanical and acoustic mass). For this reason, where confusion may arise, subscripts are used to specify domains and differentiate variables. When dealing with a single domain, subscripts may be omitted for clarity. Variables that make only minor appearances are not included in the table below, though there meaning should be clear from the text.

Symbol	Meaning	Units
General:		
f	Frequency	Hertz - [Hz]
ω	Radian frequency	Radians per second [Rad/s]
t	Time	Seconds - [s]
j	Imaginary unit	[-]
$\square^*$	Complex conjugate	[-]
$\Re(\square)$	Real part	[-]
$\Im(\square)$	Imaginary part	[-]
Electrical domain:		
V	Voltage	Volts - [V]
i	Current	Amps - [A]
Z	Impedance	Ohms - [Ω]
L	Inductance	Henry - [H]
C	Capacitance	Farads - [F]
R	Resistance	[Ω]
G	Conductance	Siemens - [S]
q	Charge	Coulombs [C]
B	Flux density	Tesla - [T]
l	Length of voice coil	Meters - [m]
Q	Q-factor	[-]
H	Transfer function	Various - [-]
Mechanical domain:		
F	Force	Newtons - Newtons - [N]
x	Displacement	Meters - [m]
u	Velocity	[m/s]
a	Acceleration	[m/s^2]
Z	Impedance	[Nm/s]
Y	Mobility	[s/Nm]
M	Mass	[kg]
C	Compliance	[m/N]
k	Stiffness	[N/m]

R	Damping factor	[Ns/m]
X	Reactance	[Ns/m]
P	Power	[J]
Acoustic domain:		
p	Pressure	[Pa]
U	Volume velocity	Volume per second - [m^3/s]
M	Mass	[??]
C	Compliance	[??]
R	Damping factor	[??]
S_D	Diaphragm area	[m^2]
ρ	Density of air	[??]
c	Speed of sound	[m/s]
V	Volume	[m^3]
a	Radius	[m]
k	Wave number	[??]
Subscripts:		
$\square_0$	Amplitude	[-]
$\square_E$	Electrical domain	[-]
$\square_M$	Mechanical domain	[-]
$\square_A$	Acoustic domain	[-]
$\square_T$	Total (inc. A, M, E)	[-]
$\square_{\square D}$	Diaphragm only	[-]
$\square_{\square S}$	Diaphragm (inc. air load)	[-]
$\square_{\square T}$	Diaphragm + enclosure	[-]
$\square_{\square f}$	Diaphragm front	[-]
$\square_{\square b}$	Diaphragm rear	[-]
$\square_{\square V}$	Vent quantity	[-]

1 Introduction

An audio signal can be readily stored, retrieved, processed, and broadcast using electronic means. However, it must be turned into an acoustic signal in order for it to be audible. The transformation of an electronic signal into audible acoustic waves is called electroacoustic transduction. A loudspeaker is such an electroacoustic transducer. The conversion from an acoustic wave to a stored audio signal is also known as electroacoustic transduction; in this case the transformation of information from the acoustic domain into an electronic representation is achieved using devices which we know as microphones.

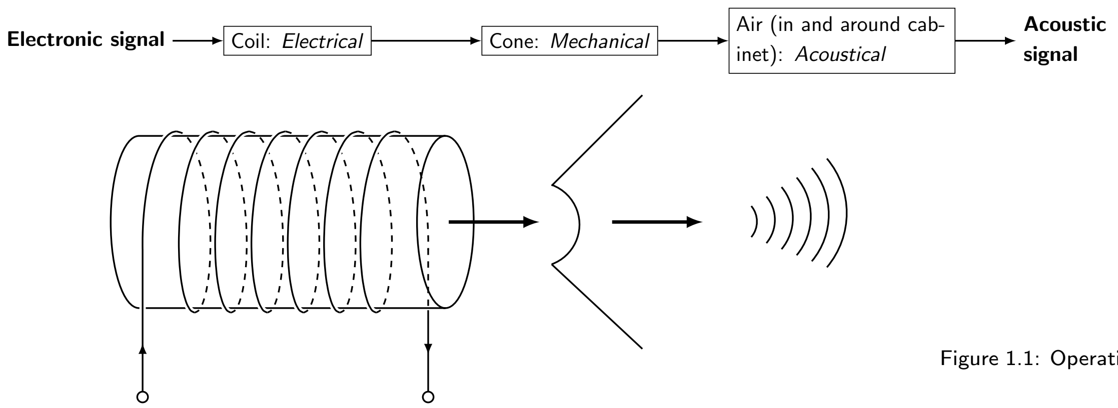


Figure 1.1: Operation of a dynamic loudspeaker.

Conversion from an acoustic wave to a varying electrical signal (or visa-versa) is not usually something that happens directly. In the case of a microphone, acoustic pressure variations are first translated into the movement of a mechanical object (the microphone diaphragm) before conversion to an electrical signal. In a loudspeaker, similarly the electrical signal is first transformed into the mechanical motion of the loudspeaker diaphragm before the motion of this physical component then impresses itself upon the surrounding air, causing acoustic disturbances. In order to fully appreciate electroacoustic transduction, we should understand the properties and limitations of both aspects; electro-mechanical transduction and mechano-acoustic transduction.

Loudspeakers and microphone are very similar in their operation; in fact, some loudspeakers can be used as effective microphones and vice-versa (a woofer can make an effective microphone for a kick drum!). Many of the design consideration of microphones and loudspeakers are similar. However, one type of loudspeaker dominates (the electrodynamic loudspeaker) while for microphones more types are commonly in use (electrodynamic/electrostatic). Therefore, in microphone design there are arguably more parameters to consider. This module will start with loudspeaker design, before moving onto microphones, though many of the same principals apply. Figure 1.1 and figure 1.2 illustrates the three domains we will consider (electrical, mechanical and acoustic) in the study of loudspeaker and microphone design. How energy is converted from domain to another has a big impact in the design of transducers.

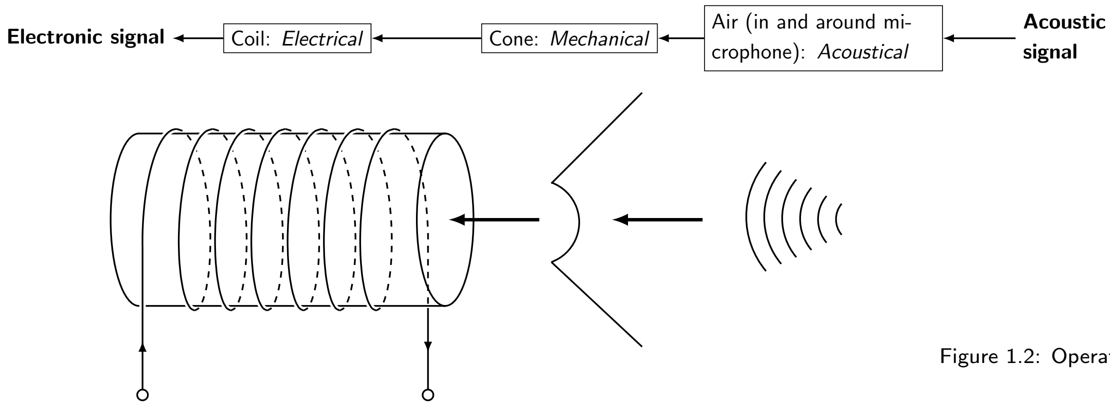


Figure 1.2: Operation of a dynamic microphone

1.1 Loudspeaker Anatomy and Operation

Electrodynamic loudspeakers were first introduced in the 1925 by Rice and Kellogg: `https://cdm16694.contentdm.oclc.org/digital/collection/p16694coll20/id/5690/`. These are often referred to as simply dynamic loudspeakers or moving-coil loudspeakers. These are probably the most popular kind of loudspeaker and will be primary focus of this module. Electrodynamic refers to the type of electro-mechanical transduction that is employed. A force needs to be applied to move the diaphragm. This force is generated by applying an electrical current to a wire coil (the voice coil), placed within a static magnetic field. When electrical current passes through a conductor in a magnetic field, it produces a force (the Lorentz force) which varies with the current applied according to the relationship,

$$F = Bli \tag{1.1}$$

where B is the magnetic flux density of the magnet, l is the total length of wire in the coil (not the physical length of the wrapped-up coil) and i is the current through the coil. In most cases in this module we will refer only to the force product Bl (motor strength) as this is only parameter of importance.

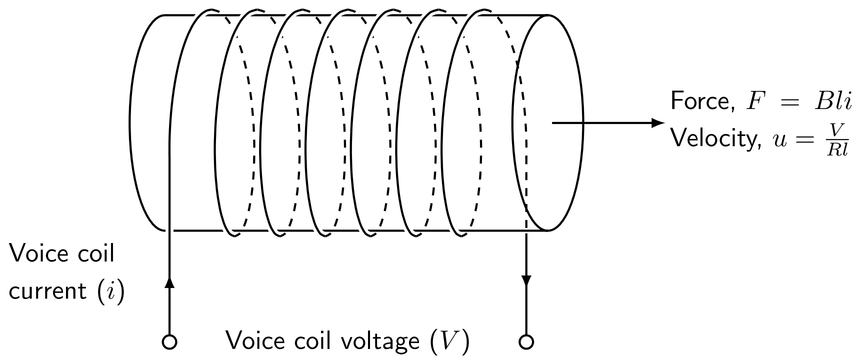


Figure 1.3: Operation of an electro-dynamic transducer.

However, an electromagnetic works both as a motor and a generator; any relative motion of the coil and magnetic field will generate a voltage at the terminals.

This is known as the back-emf (electromotive force) and represents a feedback system which makes the analysis of electrodynamic loudspeakers a little more complicated. The voltage (V) generated in a coil with velocity u is related by the following;

$$V = Blu \tag{1.2}$$

The coil and magnet assembly are known as the motor structure of the loudspeaker. Figure 1.4 shows a schematic of a typical modern dynamic loudspeaker. The principal of operation has not changed since introduced in the 1920s! The cone, connected to the voice coil, moves in and out creating fluctuations of high and low air pressure.

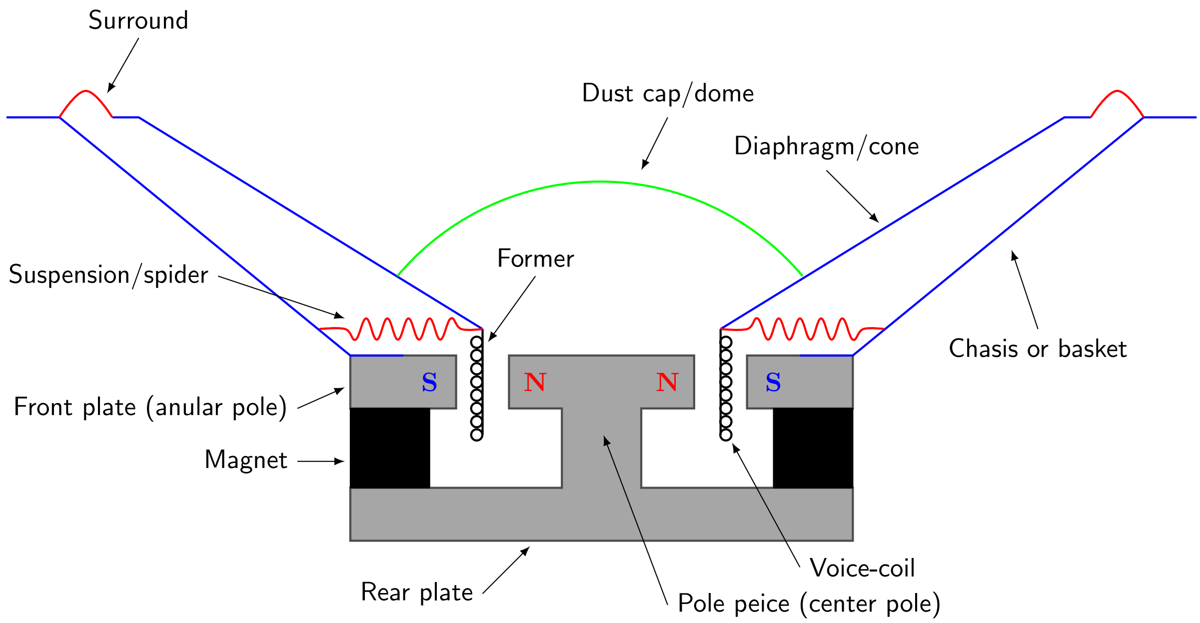


Figure 1.4: Schematic of a typical dynamic loudspeaker.

The diaphragm is attached to the basket at two points, the surround, and the spider. Together these components are known as the suspension. The spider and surround are flexible to allow for movement while preventing forms of motion that would be detrimental to the performance of the driver. The suspension is required to deliver a restoring force so that the driver returns to an equilibrium position after being displaced. The spider is intended to allow axial but prevent radial movement, by restricting the motion of the diaphragm to one axis (inwards and outwards). This allows the coil to move freely along the axis of the magnet's core (or 'pole') without touching the sides of the magnetic gap. The spider is usually made from a material such as cotton while the surround is usually made of rubber. Compromises of durability against sensitivity, or power handling against precision of response, need to be made. The air behind the dust cap will be compressed (and rarefied) when the driver is in operation and suitable vents need to be present to allow the air to escape. Care needs to be employed in the design of the vents to ensure that turbulence does not occur as air enters and escapes causing noise known as chuffing.

1.2 Modelling Approaches

To design a loudspeaker, we first need a theoretical model that can be used to predict the operation of a loudspeaker given some design parameters. We can then find the design parameters that give us a desired response. This will enable us to build a simple computer model and avoid the expense of having to build physical prototypes. Several modelling approaches are available.

1.2.1 Analytical Models

The dynamic behaviour of vibrating structures (e.g. loudspeakers) are often modelled analytically by means of differential equations, i.e. equations that relate some function with its derivatives. Although a powerful approach, the complex and multi-domain nature of loudspeakers makes the analytical formulation of a model very complicated, if not impossible.

1.2.2 Numerical Methods

The key issue with developing analytical models based on the differential equations of an entire system is that it is very difficult (if not impossible) to specify the necessary geometry/boundary conditions. One solution to this problem is to take the complex system and break it down, or discretise it, into lots of much simpler problems (so called 'finite elements') which we can then recombine later on. This idea is at the heart of the Finite Element Method (you will have the option to cover this in more detail in your final year).

This sort of approach is referred to as a numerical approach, as opposed to say an analytical or empirical approach. To do this we need to be able to assign system properties like geometry, material characteristics, etc. The implementation of the Finite Element Method is typically left to a computer program (e.g. COMSOL, ANSYS, etc.).

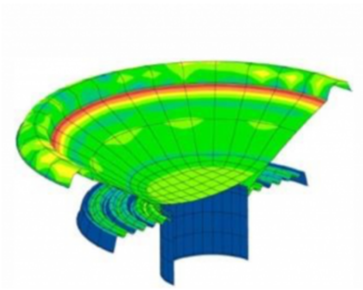


Figure 1.5: Example finite element simulation of a woofer.

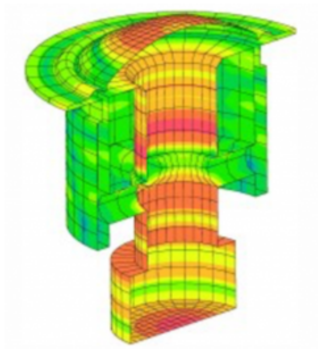


Figure 1.6: Example finite element simulation of a tweeter.

However, the translation from initial design to a working FEM model is not straight forward. It can be very difficult to set up the models correctly and take a long time to get working properly. Also, coupling the 3 domains is not trivial (need a multi-physics based FEM program), and things will almost certainly go wrong before they go right! Nevertheless, loudspeaker manufactures will typically use FEM for fine tuning designs, because when it works it is surprisingly good!

1.2.3 Lumped Parameter Modelling

By making a few simplifying assumptions we can take complex geometrical components, like diaphragms, and lump them into a single equivalent element. This enables us to simplify the problem tremendously. Unlike the FE approach, where we would have loads of small elements, now we have 1 big lumped element.

So how do we use this approach to model a loudspeaker? Well, we can take the diaphragm and model it as a single lumped mass element. We can take the suspension and the spider and model their combined effect as a single lumped spring element. Then finally we can collect together all of the damping in the system into a lumped damper or dash-pot.

2 Lumped Parameter Modelling

To treat a component as a lumped element we have to assume that its only independent variable is time, i.e. there is **no** spatial variation of any kind.

$$u_{cone} = u(x,t) = u(t) \tag{2.1}$$

Lets take as an example the diaphragm of a loudspeaker. Suppose we excite the diaphragm with some force at a very high frequency. The wave length of this excitation is so small that waves can actually travel up and down the cone and cause standing waves (think room modes but on a cone!) This is what we call cone breakup. Clearly we have some spatial distribution here, so cant use the lumped element approach. Now instead, suppose we excite the diaphragm at a

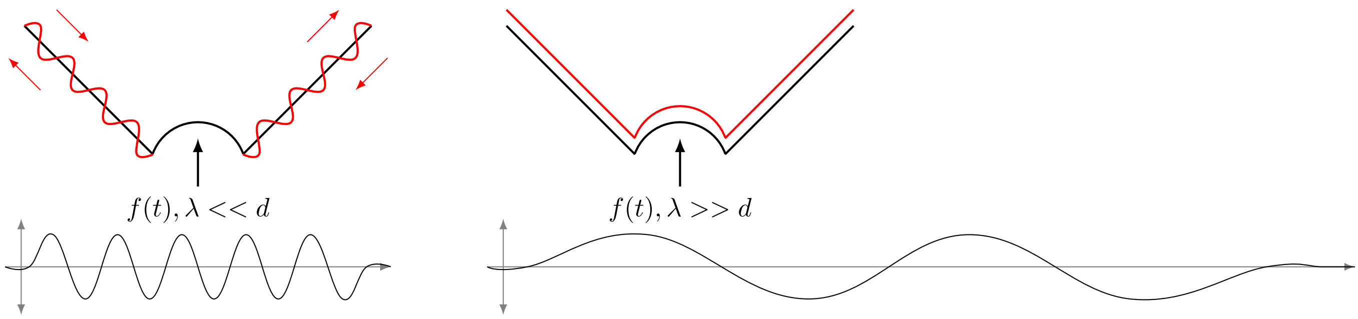


Figure 2.1: Lumped element assumption for a loudspeaker cone.

much lower frequency, so that we have a much longer wave length. In fact, the wave length is greater than the size of our diaphragm! This means that we don't get any waves travelling up and down the cone. As a result, we have no spatial distribution! And the cone moves uniformly at all points! In this case we can treat the diaphragm as a lumped element.

For a microphone we have a similar assumption, instead of an applied force, we say that there is no change in pressure or velocity across the diaphragm.

The main assumption of the lumped element approach is that there is no wave behavior in our system. This is clearly a low frequency approximation. But it's a surprisingly good one! It helps us simplify our equations of motion, and it will get us all the way through to the end of semester two!

2.1 Impedance: A Common Language

Key to the success of the lumped parameter approach is the use of a common languuage across the three domains of interest (electrical, mechanical and acoustical). This common language is called impedance.

Electrical impedance is the measure of the opposition that a circuit presents to a current when a voltage is applied, and is defined as,

$$Z_E = \frac{V}{i} \tag{2.2}$$

Mechanical impedance is a measure of how much a structure resists motion (velocity) when subjected to a force, and is defined as,

$$Z_M = \frac{F}{u} \tag{2.3}$$

Acoustic impedance is a measure of the opposition that a system presents to the acoustic flow (Volume velocity) when subjected to acoustic pressure, and is defined as,

$$Z_A = \frac{p}{U} \tag{2.4}$$

We will see later that by drawing an analogy between the electrical and mechanical/acoustic variables, we can treat the mechanical and acoustical impedance as if they were electrical, and in turn develop equivalent electrical circuits which describe the dynamic behaviour of the physical systems.

2.2 Equivalent Circuits

The primary aim of this first semester is to develop equivalent electrical circuits which accurately model the low frequency behaviour of dynamics loudspeakers. An example of such a circuit is shown in figure 2.2.

Once an equivalent circuit is available, by applying AC circuit theory we will be able to determine the frequency and efficiency characteristics of the loudspeaker. By developing appropriate sound radiation models other characteristics such as directivity will also be available.

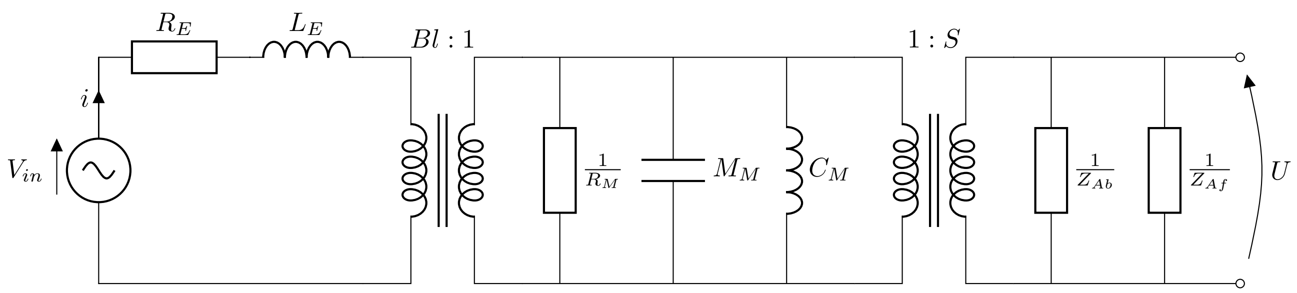


Figure 2.2: An equivalent electrical circuit for a dynamic loudspeaker

To build an equivalent circuit we must first develop electrical analogies for the mechanical and acoustical domains. These will allow us to interpret the physical domains as simple electric circuits. We will then couple these domains by adapting the theory of ideal transformers to suit electro-mechanic and mechano-acoustic transduction. Once coupled the equivalent circuit can be transformed into a simple form, and used to investigate the effect of design modifications, e.g. the introduction of a sealed or vented enclosure.

3 AC Circuit Theory

3.1 Electrical Quantities

Voltage V can be interpreted as a force that pushes electrons through some conductor. The greater the voltage, the greater the push! The difference in voltage between any two points in a circuit is called a potential difference, or voltage drop.

Current i is the continuous and uniform flow of charge carrying electrons through a conductor. The greater the current the greater the flow of electrons/charge!

Impedance Z is the capacity of a material to resist or oppose the flow of a current. The greater the impedance, the harder it is for charge to flow!

Voltage, current and impedance are related through the following equation,

$$V = iZ. \tag{3.1}$$

3.2 Kichhoff's Current Law

Kirchhoff's current law states that: '*the total current or charge entering a junction or node is exactly equal to the charge leaving the node as it has no other place to go except to leave, as no charge is lost within the node*'. This is a restatement of the principle of the **conservation of charge**.

Mathematically this law can be written as,

$$\sum_n i_n = 0 \tag{3.2}$$

where i_n is the nth current flowing into (positive i_n) or out of (negative i_n) a junction.

3.3 Kichhoff's Voltage Law

Kirchhoff's voltage law states that: '*in any closed loop network, the total voltage around the loop is equal to the sum of all the voltage drops within the same loop*' (i.e. the algebraic sum of all voltages within the loop must be equal to zero). This is a restatement of the principle of the **conservation of energy**.

Mathematically this law can be written as,

$$\sum_n V_n = 0 \tag{3.3}$$

where V_n is the nth voltage drop (negative V_n) or voltage source (positive V_n) around the closed loop.

3.4 Series and Parallel Elements

Using Kirchhoff's laws, outlined above, it is possible to determine the total impedance of a collection of arbitrary impedances. In general there exist two types of component arrangement; series or parallel, although these arrangements can be built up to form far more complex circuits.

3.4.1 Series Elements

A series arrangement of 3 arbitrary impedances is shown in figure 3.1. For a series arrangement, the total impedance Z_T presented by the circuit is,

Excercise: Can you derive this equation using Kirchhoff's laws?

$$Z_T = Z_1 + Z_2 + Z_3. \tag{3.4}$$

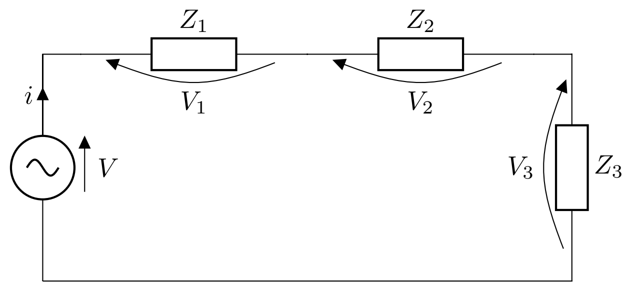


Figure 3.1: Three arbitrary impedances in series

3.4.2 Parallel Elements

A parallel arrangement of 3 arbitrary impedances is shown in figure 3.1. For a parallel arrangement, the total impedance Z_T presented by the circuit is,

Exercise: Can you derive this equation using Kirchhoff's laws?

$$Z_T = \left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right)^{-1}. \tag{3.5}$$

In the case that there are only 2 elements in parallel, equation 3.5 can be reduced

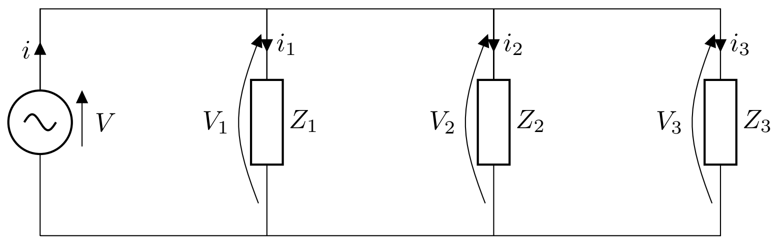


Figure 3.2: Three arbitrary impedances in parallel

to an alternate form called the *product over sum rule*,

Exercise: Can you derive this equation from equation 3.5, assuming only two elements?

$$Z_T = \frac{Z_1 Z_2}{Z_1 + Z_2}. \tag{3.6}$$

Equation 3.6 is often far more convenient than equation 3.5, especially when algebraic manipulations are required.

3.5 Detour: Complex Numbers

A complex number is a number that can be expressed in the form,

$$z = a + jb \tag{3.7}$$

where a and b are real numbers, and j is the solution of the equation $x^2 = -1$ (often the letter i is used instead, however when dealing with currents, which are also denoted by i, we use j for the imaginary unit).

Complex numbers are encountered when we use the Fourier transform to analyse problems in the frequency domain. Quantities that we measure physically (e.g. pressure, velocity, voltage, etc.) are clearly real, and have no imaginary part. When transformed into the frequency domain via the Fourier transform however, we get complex coefficients that describe the amplitude $|z|$ and relative phases θ of the signals constituent frequencies.

The representation of a complex number as a magnitude and phase angle is referred to as the polar form,

$$z = |z|e^{j\theta}. \tag{3.8}$$

The representation as given by equation 3.7 is referred to as the Cartesian form.

In both the Cartesian and polar form, a complex number is represented by a vector that lies in what is called the complex plane (or an Argand diagram). This is a 2D space where the x and y coordinates correspond to the real and imaginary components of the complex number, respectively. An example is given in figure 3.3.

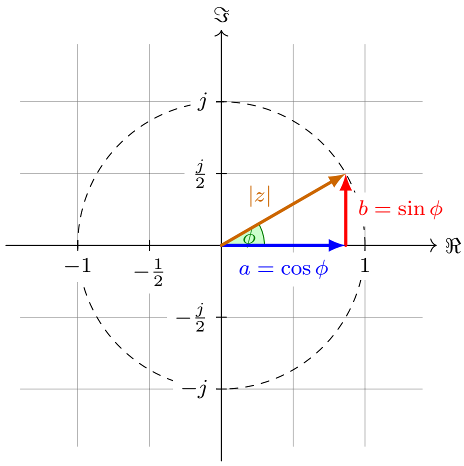


Figure 3.3: Complex amplitude of displacement represents the magnitude and phase of a particular frequency

From the complex plane we can see that the length, or magnitude, of a complex number is given simply by Pythagoras' Theorem,

$$|z| = \sqrt{a^2 + b^2}. \tag{3.9}$$

Similarly, the phase angle can be obtained as,

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \tag{3.10}$$

although care has to be take depending on which quadrant of the complex plane the vector is residing (e.g. $z = -a - jb$ will have the same phase angle as $z = a + jb$ even though they point is different directions.).

A very important equation concerning complex numbers is that of Euler's formula,

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{3.11}$$

which relates the trigonometric functions $\sin$ and $\cos$ to the complex exponential function $e^{j\theta}$, where θ is simply an appropriate phase angle given in radians.

Equation 3.11 is perhaps one of the most useful equations in all of physics and engineering. It allows us to express periodic functions (such as $\sin$ and $\cos$), and by Fourier's theorem any signal, in terms of complex exponentials. This can greatly simplify the mathematics of many problems (e.g. it is much easier to take the derivative of exponentials than trig functions!)

Listed below are some other useful identities when dealing with complex numbers:

$$j^2 = -1 \tag{3.12}$$

$$\frac{1}{j} = -j \tag{3.13}$$

$$|z| = zz^* \tag{3.14}$$

$$\Re(z) = \frac{z + z^*}{2} \tag{3.15}$$

$$\Im(z) = \frac{z - z^*}{2i} \tag{3.16}$$

where $z^* = a - jb$ is the complex conjugate of $z = a + jb$.

3.6 Electrical Impedance

Although there exist many others, in the lumped parameter/equivalent circuit modelling of a loudspeaker there are 3 key electrical components: the resistor, capacitor and inductor. It will turn out that by using just these three elements, we can model the complete low frequency behaviour of a loudspeaker. To do so we must first derive each of their electrical impedances,

$$Z_E = \frac{V}{i}. \tag{3.17}$$

3.6.1 Resistors

Resistors are the simplest electrical components available. There purpose is to limit the flow of current by presenting a resistance to its motion. The circuit diagram representation of a resistor is shown in figure 3.4. An ideal resistor has an impedance that is independent of frequency and characterised by its resistance R,

$$Z_R = R. \tag{3.18}$$

The impedance of a resistor is sometimes denoted by its reciprocal value, i.e. the conductance

$$G = \frac{1}{R}. \tag{3.19}$$

A resistor's opposition to the flow of current generates heat, i.e. it converts electrical energy into heat energy.

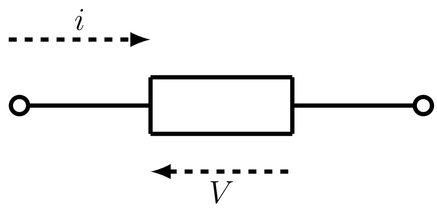


Figure 3.4: Symbol for electrical resistor

3.6.2 Capacitors

A capacitor is a passive electrical component. Unlike a resistor however, a capacitor has the ability, or the capacity, to store energy. It does this in the form of an electrical charge, which creates a potential different across its two ends. The circuit diagram representation of a capacitor is shown in figure 3.5.

The capacitance C of a capacitor is what relates the voltage drop across its two ends to the charge q,

$$q = CV. \tag{3.20}$$

Noting that current is defined as the time rate of change of charge.

$$i = \frac{dq}{dt} = C\frac{dV}{dt} \tag{3.21}$$

Now if we assume the element is being driven by a periodic current/voltage, such that $i = i_0 e^{j\omega t}$ and $V = V_0 e^{j\omega t}$, the above becomes,

$$i = C\frac{dV}{dt} = j\omega CV \tag{3.22}$$

from which can find the impedance as

$$Z_C = \frac{1}{j\omega C}. \tag{3.23}$$

Equation 3.23 indicates that the electrical impedance of a capacitor is not only inversely proportional to frequency, but complex. Note that complex impedance is an indication of *energy storage*.

Using the impedance derived above we can express the voltage across a capacitor in the form,

$$V_0 e^{j\omega t} = \frac{-j}{\omega C} i_0 e^{j\omega t} \tag{3.24}$$

where the identity $\frac{1}{j} = -j$ has been used. Noting that $j = e^{j\frac{\pi}{2}}$ we can rewrite this as,

$$V_0 e^{j\omega t} = \frac{1}{\omega C} i_0 e^{j\omega t} e^{-j\frac{\pi}{2}} = \frac{1}{\omega C} i_0 e^{j(\omega t - \frac{\pi}{2})}. \tag{3.25}$$

According to equation 3.25 the current through a capacitor lags behind the voltage by $\theta = -\frac{\pi}{2}$.

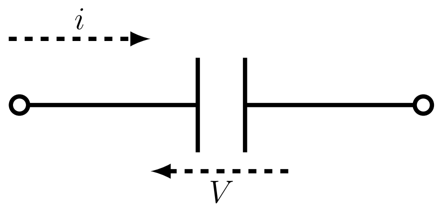


Figure 3.5: Symbol for electrical capacitor

3.6.3 Inductors

An inductor is passive electrical component consisting of a coil of wire which is designed to take advantage of the relationship between magnetism and electricity as a result of an electric current passing through the coil. Like a capacitor, an inductor can store energy. Unlike a capacitor however, an inductor stores this energy as a magnetic field, as opposed to an electrical one. The circuit diagram representation of an inductor is shown in figure 3.6.

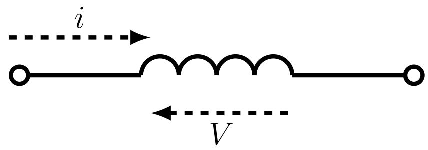


Figure 3.6: Symbol for electrical inductor

The inductance L of an inductor relates the time rate of change of current through the element to the voltage drop across its terminals,

$$V = L\frac{di}{dt} \tag{3.26}$$

Again, if we assume the element is being driven by a periodic current/voltage, such that $i = i_0 e^{j\omega t}$ and $V = V_0 e^{j\omega t}$, the above becomes,

$$V = j\omega Li \tag{3.27}$$

from which we obtain the impedance as,

$$Z_L = j\omega L. \tag{3.28}$$

Equation 3.28 indicates that the electrical impedance of a capacitor is not only proportional to frequency, but complex (i.e. the inductor stores energy).

Using the impedance derived above we can express the voltage across an inductor in the form,

$$V_0 e^{j\omega t} = j\omega L i_0 e^{j\omega t}. \tag{3.29}$$

Again, noting that $j = e^{j\frac{\pi}{2}}$ we can rewrite this as,

$$V_0 e^{j\omega t} = \omega L i_0 e^{j\omega t} e^{j\frac{\pi}{2}} = \omega L i_0 e^{j(\omega t + \frac{\pi}{2})}. \tag{3.30}$$

According to equation 3.30 the current through an inductor leads the voltage by $\theta = \frac{\pi}{2}$.

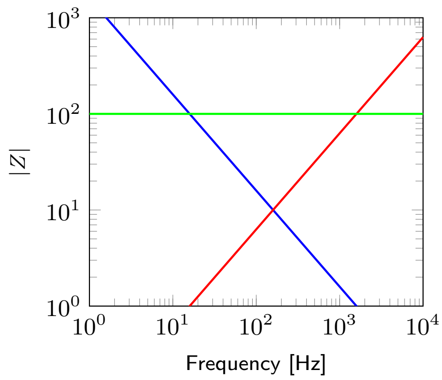
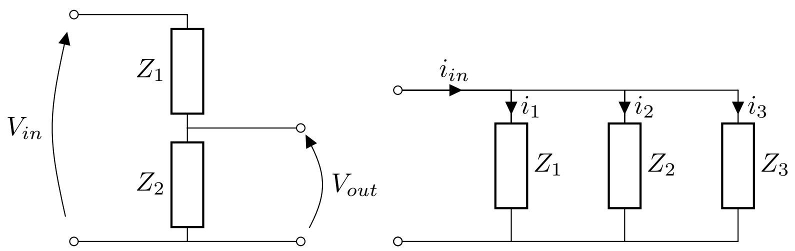


Figure 3.7: Impedance vs. frequency characteristics of a resistor (green), capacitor (blue), and inductor (red).

3.7 Divider Circuits

By arranging collections of electrical components we can create circuits that manipulate the voltage and/or current they are being driven with. Two common types are shown in figure 3.8; the potential (voltage) divider and the current divider.



(a) Potential divider (b) Current divider

Figure 3.8: Voltage and current divider circuits

3.7.1 Voltage Divider

The voltage across the input terminal of a potential divider circuit depends on the total impedance of the circuit,

$$V_{in} = iZ_T. \tag{3.31}$$

From the above we can determine the total current flowing through the circuit as,

$$i = \frac{V_{in}}{Z_T}. \tag{3.32}$$

The voltage across the output terminal of a potential divider circuit depends on the impedance of the output component,

$$V_{out} = iZ_{out}. \tag{3.33}$$

Note that the same current flows through both elements. Substituting in the current we arrive at the equation for a voltage divider,

$$V_{out} = \frac{Z_{out}}{Z_T} V_{in}. \tag{3.34}$$

In the special case that just two elements are present, and the output is being taken over the second element, the output voltage is given by,

$$V_{out} = \frac{Z_2}{Z_1 + Z_2} V_{in}. \tag{3.35}$$

The transfer function (or gain) of this circuit is given by,

$$H = \frac{V_{out}}{V_{in}} = \frac{Z_2}{Z_1 + Z_2}. \tag{3.36}$$

3.7.2 Current Divider

The total current flowing through a current divider depends on the total impedance of the circuit,

$$V = i_T Z_T. \tag{3.37}$$

The current flowing though branch n is proportional to the impedance of that branch,

$$i_n = \frac{V}{Z_n}. \tag{3.38}$$

Now by substituting in for the voltage we arrive at the equation for a current divider,

$$i_n = \frac{Z_T}{Z_n} i_T. \tag{3.39}$$

In the special case that just two elements are present, and the output is taken through the second branch, the output current is given by,

$$i_{out_2} = \frac{Z_1}{Z_1 + Z_2} i_{in}. \tag{3.40}$$

The transfer function (or gain) of this circuit is given by,

$$H = \frac{i_{out_2}}{i_{in}} = \frac{Z_1}{Z_1 + Z_2}. \tag{3.41}$$

3.8 Filter Circuits

When analysing filter circuits we are typically interested in their transfer functions, i.e. their gain response as a function of frequency. Depending on whether it is the voltage of current that is of interest, the respective transfer functions is defined as so,

$$H_V = \frac{V_{out}}{V_{in}} \tag{3.42}$$

$$H_i = \frac{i_{out}}{i_{in}}. \tag{3.43}$$

In what follows we will analyse the voltage transfer function of 3 different filter circuits, a series RC, RL and RLC.

3.8.1 RC Circuit

An AC voltage is applied to the circuit. Voltage across the resistor is $v_r = iR$ with R being constant. Voltage across the capacitor is $v_c = \frac{1}{C}\int i\ dt$ where C is the constant capacitance.

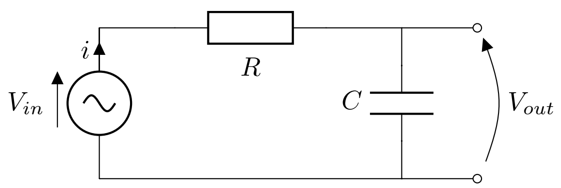


Figure 3.9: RC circuit

According to Kirchhoff's law the applied voltage is the sum of the other voltages in the circuit.

$$V_{in} = V_r + V_c \tag{3.44}$$

We could form a differential equation and solve this equation. An easier way to analyse this circuit is to treat it as a potential divider. If the output is taken across the capacitor the output voltage V_c is,

$$V_{out} = \frac{Z_C}{Z_R + Z_C} V_{in}. \tag{3.45}$$

Given what we know about the impedance of capacitors and resistors, we can express the transfer function or gain (output divided by input) as,

$$\frac{V_{out}}{V_{in}} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}. \tag{3.46}$$

We can tidy this up by multiplying top and bottom by $j\omega C$,

$$\frac{V_{out}}{V_{in}} = \frac{1}{j\omega RC + 1}. \tag{3.47}$$

Equation 3.47 is the transfer function of our RC circuit. It is often helpful to look at the magnitude of the gain,

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1 + (\omega RC)^2}}. \tag{3.48}$$

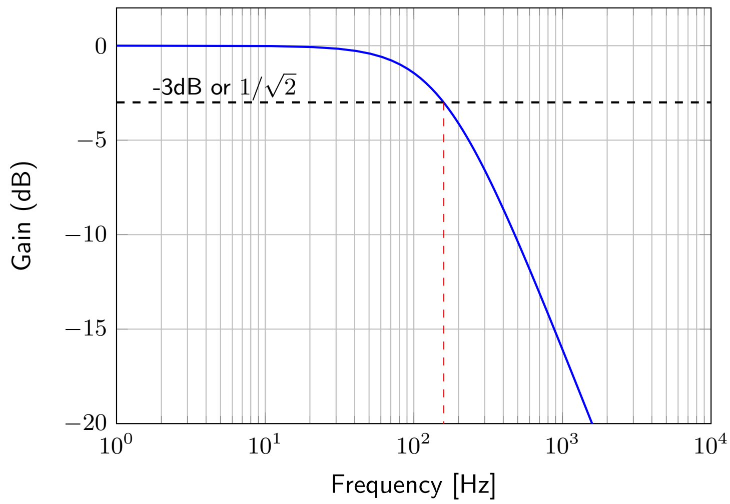


Figure 3.10: Gain response for an RC circuit, output taken across capacitor. $C = 1 \times 10^{-5}$ F, $R = 100\ \Omega$, and $f_c = \frac{1}{2\pi}\frac{1}{RC} = 159$ Hz

To establish what sort of filter figure 3.9 represents we can look at the values of its magnitude as frequency tends to the extreme limits of $\omega \to 0$ and $\omega \to \infty$. In the limit that $\omega \to 0$ (i.e. at low frequencies) we have that

$$\left|\frac{V_{out}}{V_{in}}\right| \to 1. \tag{3.49}$$

In the limit that $\omega \to \infty$ (i.e. at high frequencies) we have that,

$$\left|\frac{V_{out}}{V_{in}}\right| \to 0. \tag{3.50}$$

The above trends are that of a low pass filter.

The cut-off frequency (ω_c) of a filter is the frequency where the power output is half that of the input. Half power implies that the gain is $1/\sqrt{2}$ or -3dB,

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1 + (\omega_c RC)^2}}. \tag{3.51}$$

Looking at 3.51 we can see this will happen when $(\omega RC)^2 = 1$. Rearranging this, we can see that,

$$\omega_c = \frac{1}{RC} \tag{3.52}$$

Having established an equation for the cut-off frequency ω_c, equation 3.47 can be parametrised using this relationship,

$$\frac{V_{out}}{V_{in}} = \frac{1}{j\frac{\omega}{\omega_c} + 1}. \tag{3.53}$$

We can also get an understanding of the phase shift introduced by the circuit by looking at 3.53. The gain at cut off (when $\omega = \omega_c$) is

$$\frac{V_{out}}{V_{in}} = \frac{1}{j+1}. \tag{3.54}$$

By multiplying the top and bottom of equation 3.54 by the complex conjugate of the denominator we can obtain,

$$\frac{V_{out}}{V_{in}} = \frac{1}{j+1}\frac{1-j}{1-j} = \frac{1-j}{2} = \frac{1}{2} - \frac{j}{2}. \tag{3.55}$$

The phase of the response can then be found by considering the Argand diagram. Using trigonometry, the phase shift at cut-off is,

$$\phi = \tan^{-1}\left(\frac{-0.5}{0.5}\right) \rightarrow -45°. \tag{3.56}$$

3.8.2 LR Circuit

An inductor in series with a resistor forms another common electrical circuit.

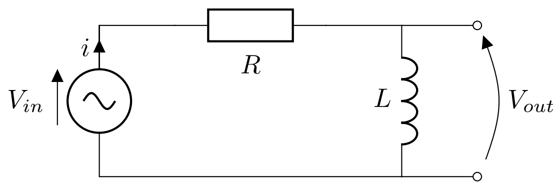


Figure 3.11: RL circuit

We can analyse this circuit in the same manner as above, using the voltage divider rule. The voltage across the inductor is,

$$V_{out} = V_{in} \times \frac{Z_L}{Z_R + Z_L}. \tag{3.57}$$

Substituting in the impedance of the resistor and inductor, the circuit gain (output divided by input) is given by,

$$\frac{V_{out}}{V_{in}} = \frac{j\omega L}{R + j\omega L} \tag{3.58}$$

which simplifies to,

$$\frac{V_{out}}{V_{in}} = \frac{1}{\frac{R}{j\omega L} + 1} \tag{3.59}$$

Taking the magnitude of equation 3.59 we have,

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{\left(\frac{R}{\omega L}\right)^2 + 1}}. \tag{3.60}$$

To establish what sort of filter figure 3.11 represents we can look at the values of its magnitude as frequency tends to the extreme limits of $\omega \rightarrow 0$ and $\omega \rightarrow \infty$. In the limit that $\omega \rightarrow 0$ (i.e. at low frequencies) we have that,

$$\left|\frac{V_{out}}{V_{in}}\right| \rightarrow 0. \tag{3.61}$$

In the limit that $\omega \to \infty$ (i.e. at high frequencies) we have that,

$$\left|\frac{V_{out}}{V_{in}}\right| \to 1. \tag{3.62}$$

The above trends are that of a high pass filter.

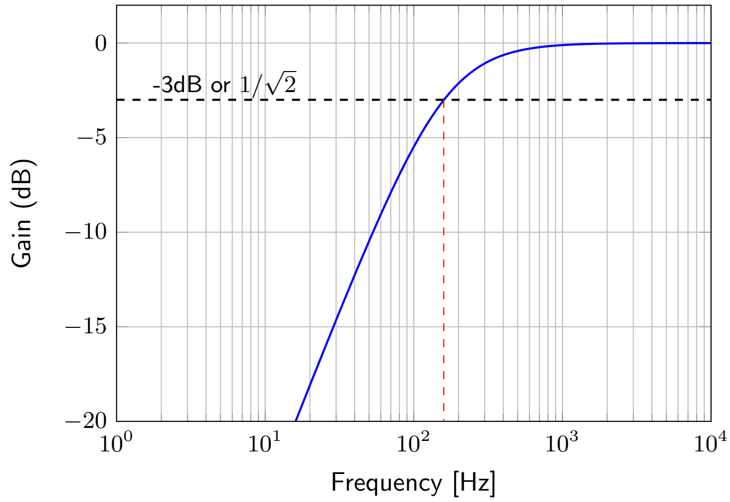


Figure 3.12: Gain response for an RL circuit, output taken across inductor. $L = 1 \times 10^{-1}$ H, $R = 100\ \Omega$, and $f_c = \frac{1}{2\pi}\frac{R}{L} = 159$ Hz

From equation 3.60 the cut-off frequency (i.e. when the gain is -3dB or $1/\sqrt{2}$ will happen when $\left(\frac{R}{\omega L}\right)^2 = 1$. Rearranging this we have that,

$$\omega_c = \frac{R}{L}. \tag{3.63}$$

Using equation 3.63, equation 3.59 can be parametrised as so (noting that $1/j = -j$),

$$\frac{V_{out}}{V_{in}} = \frac{1}{1 - j\frac{\omega_c}{\omega}}. \tag{3.64}$$

We can also get an understanding of the phase shift introduced by the circuit by looking at 3.64. The gain at cut off (when $\omega = \omega_c$) is

$$\frac{V_{out}}{V_{in}} = \frac{1}{1 - j}. \tag{3.65}$$

By multiplying the top and bottom of equation 3.65 by the complex conjugate of the denominator we can obtain,

$$\frac{V_{out}}{V_{in}} = \frac{1}{1-j}\frac{1+j}{1+j} = \frac{1+j}{2} = \frac{1}{2} + \frac{j}{2}. \tag{3.66}$$

The phase of the response can then be found by considering the Argand diagram. Using trigonometry, the phase shift at cut-off is,

$$\phi = \tan^{-1}\left(\frac{0.5}{0.5}\right) \to 45°. \tag{3.67}$$

3.8.3 LCR Circuit

An inductor in series with a resistor and a capacitor forms a resonating circuit.

The total impedance this circuit is,

$$Z_T = R + j\omega L + \frac{1}{j\omega C} \tag{3.68}$$

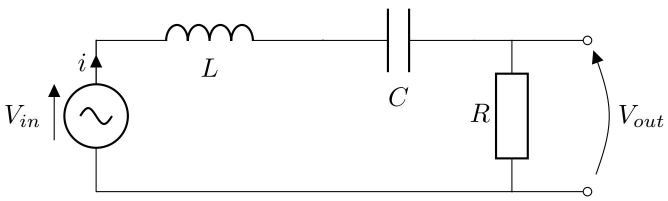


Figure 3.13: LCR circuit

We can analyse this circuit in the same manner as above using the voltage divider rule. The voltage across the resistor is,

$$V_{out} = V_{in} \times \frac{Z_R}{Z_R + Z_L + Z_C}. \tag{3.69}$$

Substituting in the components' impedance the gain (output divided by input) is given by,

$$\frac{V_{out}}{V_{in}} = \frac{R}{R + j\omega L + \frac{1}{j\omega C}}. \tag{3.70}$$

Dividing top and bottom by $1/R$, the above simplifies to,

$$\frac{V_{out}}{V_{in}} = \frac{1}{1 + \frac{j\omega L}{R} + \frac{1}{j\omega RC}}. \tag{3.71}$$

Taking the magnitude of equation 3.71 then yields,

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1 + \left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)^2}}. \tag{3.72}$$

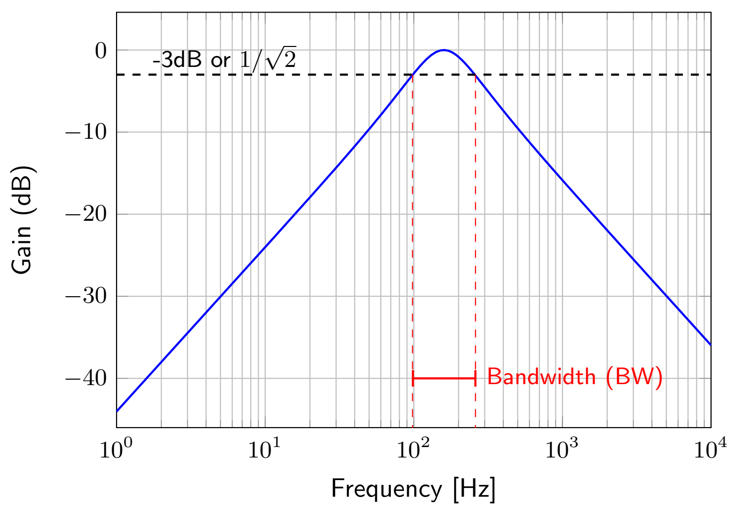


Figure 3.14: Gain response for an LRC circuit, output taken across resistor. $C = 1 \times 10^{-5}$ F, $L = 1 \times 10^{-1}$ H, $R = 100\ \Omega$, and $f_r = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}$.

To establish what sort of filter figure 3.13 represents we can look at the values of its magnitude as frequency tends to the extreme limits of $\omega \to 0$ and $\omega \to \infty$. In the limit that $\omega \to 0$ (i.e. at low frequencies) we have that,

$$\left|\frac{V_{out}}{V_{in}}\right| \to 0 \quad \text{(Determined by capacitance).} \tag{3.73}$$

In the limit that $\omega \to \infty$ (i.e. at high frequencies) we have that,

$$\left|\frac{V_{out}}{V_{in}}\right| \to 0 \quad \text{(Determined by inductance).} \tag{3.74}$$

Unlike the RC and RL circuit considered above, there exists an intermediate frequency (between 0 and ∞) where the gain is a maximum. This frequency corresponds to when the reactance is zero,

$$\frac{j\omega L}{R} - \frac{j}{\omega RC} = 0 \quad \text{or} \quad \left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right) = 0 \tag{3.75}$$

By rearranging equation 3.75 we can get an expression for the resonant frequency of the circuit,

$$\omega_r = \frac{1}{\sqrt{LC}} \tag{3.76}$$

The above trends are that of a band pass filter, whose centre frequency is ω_r.

Plotting 3.71 we get the response in Figure 3.14. Here we can see that the resonant peak has a particular width, or bandwidth (BW). The BW and resonant frequency determine a useful parameter known as the Q-Factor or quality factor. This is the ratio to the resonant frequency to the bandwidth. In this circuit the bandwidth is $BW = R/L$ hence,

$$Q = \frac{\omega_r}{BW} = \frac{L}{R\sqrt{LC}} = \frac{1}{R}\sqrt{\frac{L}{C}}. \tag{3.77}$$

3.9 Norton and Thevenin's Theorems

Norton's and Thevenin's theorems are a pair of very useful theorems for simplifying AC circuits. They let allow us to introduce ideal voltage and current sources in place of complex networks of electrical components. The two theorems are in fact deeply related and are the dual of one another.

3.9.1 Norton's Theorem

Norton's theorem states that '*any linear electrical network that contains only voltage sources, current sources and impedances can be replaced by an ideal current source i_{no} in parallel with an appropriate impedance Z_{no}*'.

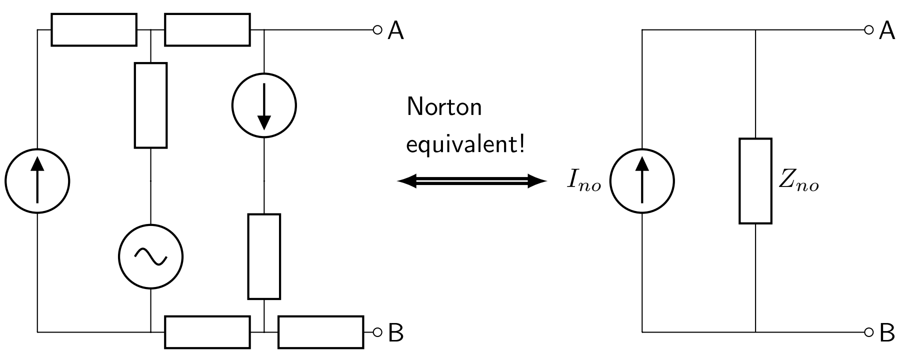


Figure 3.15: Norton equivalent circuit.

The procedure for applying Norton's theorem is as follows:

- The equivalent current i_{no} is the current obtained at terminals A B of the network with terminals A B short circuited.
- The equivalent impedance Z_{no} is the impedance obtained at terminals A B of the network with all its voltage sources short circuited and all its current sources open circuited.

Consider the following example. An ideal voltage source V_S is coupled to a network of impedance elements, which themselves are connected to a load impedance Z_L. Norton's Theorem can be used to replace the complex network of impedances with a single equivalent impedance in parallel with an equivalent Norton voltage source.

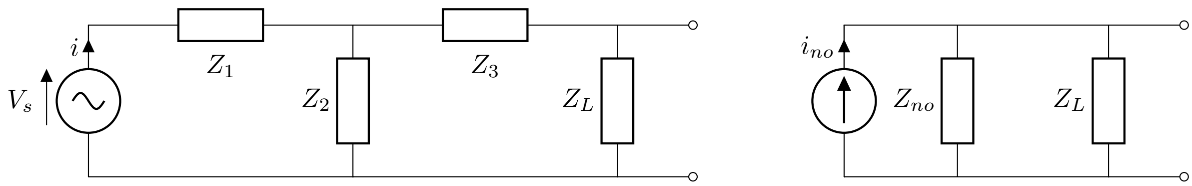


Figure 3.16: Example Norton Theorem

To find the equivalent Norton current we begin by replacing the load impedance with a short circuit. The total current running through the circuit is given by,

$$i_T = \frac{V_s}{Z_T} \tag{3.78}$$

where Z_T is the total impedance of the circuit,

$$Z_T = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}. \tag{3.79}$$

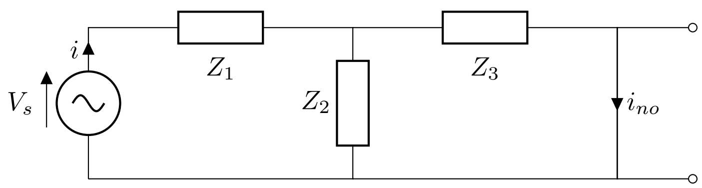


Figure 3.17: Example - finding the Norton current.

Using the current divider equation (3.40) the Norton current is given by,

$$i_{no} = i_3 = \frac{Z_2}{Z_2 + Z_3} i_T = \frac{Z_2}{Z_2 + Z_3} \frac{V_s}{Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}}. \tag{3.80}$$

To find the equivalent Norton impedance we begin by replacing the voltage source with a short circuit and removing the load impedance. The impedance of the remaining circuit, across the load terminals, is obtained as,

$$Z_{no} = Z_3 + \frac{Z_1 Z_2}{Z_1 + Z_2}. \tag{3.81}$$

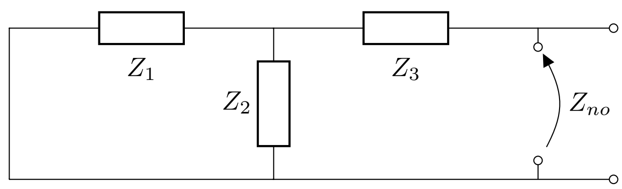


Figure 3.18: Example - finding the Norton impedance.

Together equations 3.80 and 3.81, representing the Norton current and impedance, respectively, provide an entirely equivalent source description (when arranged in parallel) from the perspective of the load impedance, as the original network.

3.9.2 Thevenin's Theorem

Thevenin's theorem states that '*any linear electrical network that contains only voltage sources, current sources and impedances can be replaced by an equivalent combination of an ideal voltage source* V_{th} *in series with an appropriate impedance* Z_{th}.'

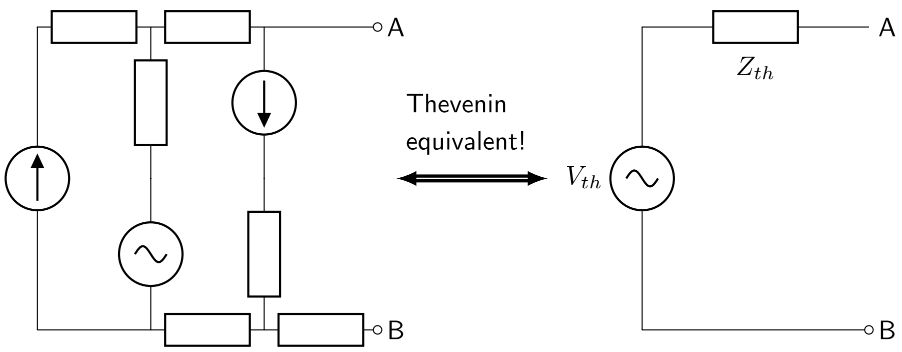


Figure 3.19: Thevenin equivalent circuit.

The procedure for applying Thevenin's theorem is as follows:

- The equivalent voltage V_{th} is the voltage obtained at terminals A B of the network with terminals A B open circuited.
- The equivalent impedance Z_{th} is the impedance that the circuit between terminals A B would have if all ideal voltage sources (e.g. V_S below) in the circuit were replaced by a short circuit and all ideal current sources were replaced by an open circuit , i.e. with the load removed (same as Norton's $Z_{no} = Z_{th}$).

Exercise: Can you find the Thevenin voltage and impedance for figure 3.16?

3.10 Drop and Flow Quantities

We have covered 3 key electrical components: the resistor, capacitor and inductor. For each component we have three generic quantities:

- The (voltage) drop across the (electrical) component
- The (current) flow through the (electrical) component
- The magnitude of the (electrical) component itself (resistance, capacitance, inductance)

These generic quantities are not limited to electrical components however. We can find analogous quantities for mechanical and acoustic systems! For a mechanical system we have force (F) and velocity (u). For an acoustic system we have pressure (p) and volume velocity (U).

But which quantity is the drop and which is the flow? The choice is ours, and it will depend on the problem at hand.

4 Mechanical Domain

For simple linear systems (like our loudspeaker systems) there are 3 main mechanical components: mass elements, springs, and dampers or dashpots. By combining these elements in different configurations we can create simple dynamic models that represent the underlying physics of real structures.

Also, it turns out that the equations which govern these mechanical elements, are very similar to those of the electrical components we have discussed so far. This similarity will enable us to interpret these mechanical systems using an '*equivalent electrical circuit*' approach.

4.1 Mechanical Impedance and Mobility

Let us begin by first recalling our definition of electrical impedance: it is the voltage across a component divided by the current flowing through it, i.e. drop divided by flow,

$$Z_E = \frac{V}{i}. \tag{4.1}$$

In a mechanical system we have a similar quantity which we call the mechanical impedance. It is defined as force over velocity,

$$Z_M = \frac{F}{u}. \tag{4.2}$$

The mechanical impedance of a structure describes the opposition that the structure presents to an applied motion. Its reciprocal value, the mobility, is used to describe the freedom of motion that a structure has,

$$Y_M = \frac{u}{F}. \tag{4.3}$$

In what follows we will derive the impedance of the 3 main mechanical components.

4.2 Mass Element

To derive the impedance of a mass element we start with Newton's 2nd Law,

$$F = Ma \tag{4.4}$$

where F is an externally applied force, M is the mass of the element, and a its acceleration. We will assume that the applied force is periodic, and so takes the form $F = F_0 e^{j\omega t}$. Clearly, the acceleration response will also be periodic, $a = a_0 e^{j\omega t}$.

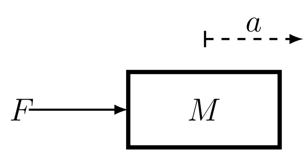


Figure 4.1: Mechanical mass element.

Substituting the acceleration for the time derivative of the mass velocity, noting that $\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t}$, leads to,

$$F = M\frac{du}{dt} = j\omega Mu. \tag{4.5}$$

The impedance is then obtained by dividing both sides by velocity,

$$Z_M = \frac{F}{u} = j\omega M. \tag{4.6}$$

Equation 4.6 indicates that the impedance of a mass element is not only proportional to frequency, but complex.

4.3 Spring Element

To derive the impedance of a spring element we start with Hooke's Law,

$$F = kx = \frac{1}{C}x \tag{4.7}$$

which states that the force exerted by a spring is linearly related to its extension x by the stiffness coefficient k. Although it is standard practice in most areas of mechanics to use the stiffness k, it is convenient when designing loudspeaker systems to use its reciprocal value, the compliance $C = \frac{1}{k}$.

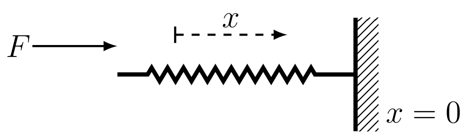


Figure 4.2: Mechanical spring element.

We will again assume that the exerted force and spring's extension are periodic, such that $F = F_0 e^{j\omega t}$ and $x = x_0 e^{j\omega t}$. Substituting the displacement for the time integral of the spring velocity, noting that $\int e^{j\omega t} = \frac{1}{j\omega}e^{j\omega t}$, leads to,

$$F = \frac{1}{C}\int u dt = \frac{1}{j\omega C}u \tag{4.8}$$

from which the spring element's impedance can be found,

$$Z_C = \frac{F}{u} = \frac{1}{j\omega C}. \tag{4.9}$$

Equation 4.9 indicates that the impedance of a spring element is also complex, but unlike the mass element, it is inversely proportional to frequency.

4.4 Damping Element

The damping mechanisms that occurs in real structures are often complex and a rigours treatment beyond the scope of this module. However, for most practical purposes (e.g. for the damping present in a loudspeaker systems) a reasonable model can be obtained using a viscous damping element, as illustrated in figure 4.3. The governing equation for a damper of this sort is,

$$F = Ru \tag{4.10}$$

where F is the reaction force exerted by the damper when driven at a velocity u. The constant of proportionality R is called the damping coefficient.

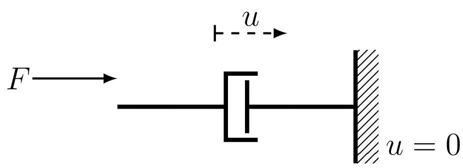


Figure 4.3: Mechanical damping element.

The impedance of a damping element is obtained straightforwardly as,

$$Z_R = \frac{F}{u} = R. \tag{4.11}$$

Equation 4.11 indicates that the impedance of a damping element is independent of frequency, and real.

4.5 Impedance Analogy

Those with a keen eye may have noticed that the electrical and mechanical impedances we have derived are of a similar form... In particular: the impedance of an electrical inductor is of the same form as the mechanical mass element,

$$Z_{EL} = j\omega L \quad \sim \quad Z_{MM} = j\omega M \tag{4.12}$$

the impedance of an electrical capacitor is of the same form as the mechanical spring element,

$$Z_{EC} = \frac{1}{j\omega C} \quad \sim \quad Z_{MS} = \frac{1}{j\omega C} \tag{4.13}$$

and the impedance of an electrical resistor is of the same form as the mechanical damping element,

$$Z_{ER} = R \quad \sim \quad Z_{MR} = R. \tag{4.14}$$

It is the above similarities that motivate the so called *impedance analogy*. According to the impedance analogy we can make the following equivalences:

$$F \leftrightarrow V \qquad \text{(Drop)} \tag{4.15}$$

$$u \leftrightarrow i \qquad \text{(Flow)} \tag{4.16}$$

That is, we can treat mechanical force F as being equivalent to electrical voltage V, and mechanical velocity as equivalent to electrical current i. By drawing this particular analogy we preserve the analogy between mechanical and electrical impedance,

$$Z_M \leftrightarrow Z_E \tag{4.17}$$

but as we will see shortly, the topology of our problem is lost. i.e. mechanical system is arranged differently to its analogous electrical circuit.

4.5.1 Mass on a Spring

To demonstrate the impedance analogy we will consider its application to a mass-spring-damper system terminated by a rigid foundation (see figure 4.4).

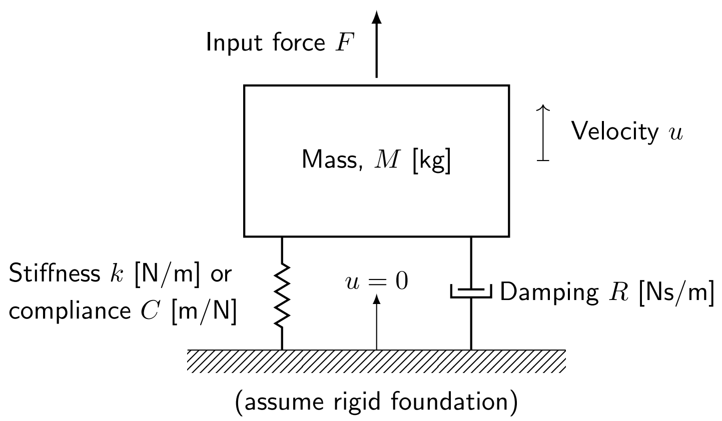


Figure 4.4: Lumped parameter mechanical model of a loudspeaker

Note that the spring and damper are both terminated at one end by a rigid foundation (i.e. $x_2 = u_2 = 0$). Consequently, their respective velocities are the same as that of the mass element, u. According to the impedance analogy, this shared velocity is equivalent to an electrical current (see equation 4.16).

Recalling that the electrical analogies for a mass, spring and damper are the inductor, capacitor and resistor, we are interested in a circuit whose components share the same current; this is achieved by the series circuit in figure 4.5.

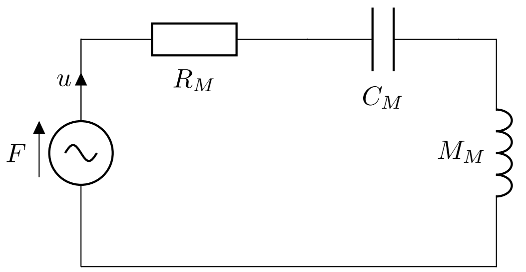


Figure 4.5: Mechanical impedance analogue.

Note that the mechanical system in figure 4.4 is excited externally by an applied force F. According to the impedance analogy $F \to V$ and so our equivalent circuit is driven by an ideal voltage source.

The force across each mechanical component is now represented by a voltage drop across the equivalent electrical component. According to Kirchhoff's voltage law we have that,

$$F - F_R - F_C - F_M = 0 \tag{4.18}$$

where F_R, F_C and F_M are the force (voltage) drops across the damper, spring and mass elements. Equation 4.18 is simply a statement of Newton's second law; the sum of forces equals mass times acceleration, $F_M = ma = \sum F$.

From the equivalent circuit in figure 4.5 we can determine the total electrical impedance (which is analogous to the mechanical impedance according to the impedance analogy) as,

$$Z_E = j\omega M_M + \frac{1}{j\omega C_M} + R_M = Z_M. \tag{4.19}$$

This is exactly what would be obtained if a more conventional analysis were undertaken (e.g. a force balance to get a differential equation).

From the above it is clear that determining the velocity of the mass element is equivalent to find the current in the equivalent circuit,

$$i = \frac{V}{Z_E} \leftarrow u = \frac{F}{j\omega M_M + \frac{1}{j\omega C_M} + R_M}. \tag{4.20}$$

In summary, the equivalent electrical circuit in figure 4.5 is entirely analogous to the mechanical system shown in figure 4.4. However, it is not the *only* analogous electrical circuit available.

4.6 Mobility Analogy

It is important to note that the idea of drawing an analogy between the electrical and mechanical domain is nothing but acknowledging a symmetry in the form of the underlying equations, and that more than one analogy may exist.

An alternative analogy is the so called 'mobility' analogy. The motivation behind the impedance analogy was to draw an equivalence between *mechanical* and electrical impedance. The mobility analogy instead draws an equivalence between mechanical *mobility* and electrical impedance.

$$Y_M \leftrightarrow Z_E \quad \text{or} \quad Z_E \leftrightarrow \frac{1}{Z_M} \tag{4.21}$$

In doing so we assume following equivalences:

$$u \leftrightarrow V \qquad \text{(Drop)} \tag{4.22}$$

$$F \leftrightarrow i \qquad \text{(Flow)} \tag{4.23}$$

According to this mobility analogy; the impedance of an electrical inductor is of the same form as the mobility of a mechanical spring element,

$$Z_{EL} = j\omega L \quad \sim \quad Y_{MS} = j\omega C = \frac{1}{Z_{MS}} \tag{4.24}$$

the impedance of an electrical capacitor is of the same form as the mobility of a mechanical mass element,

$$Z_{EC} = \frac{1}{j\omega C} \quad \sim \quad Y_{MM} = \frac{1}{j\omega M} = \frac{1}{Z_{MM}} \tag{4.25}$$

and the impedance of an electrical resistor is of the same form as the mobility of a mechanical damping element,

$$Z_{ER} = R \quad \sim \quad Y_{MR} = \frac{1}{R} = G = \frac{1}{Z_{MR}} \tag{4.26}$$

It is the above similarities that motivate the so called *mobility analogy*.

Although the mobility analogy may seem less intuitive, it does makes some physical sense, from a measurement perspective. To measure force we have to interrupt the mechanical system – to measure current we have to interrupt the electric circuit. Similarly, velocity can be measured without interruption – voltage can be measured without interruption.

4.6.1 Mass on a Spring

To demonstrate the mobility analogy we will consider its application to the same mass-spring-damper system as in figure 4.4.

According to the mobility analogy, the shared velocity of the mass, spring and damper is equivalent to an electrical voltage (see equation 4.23). Recalling that the electrical (mobility) analogies for a mass, spring and damper are the capacitor, inductor and resistor, we are interested in a circuit whose components share the same voltage; this is achieved by the parallel circuit in figure 4.6.

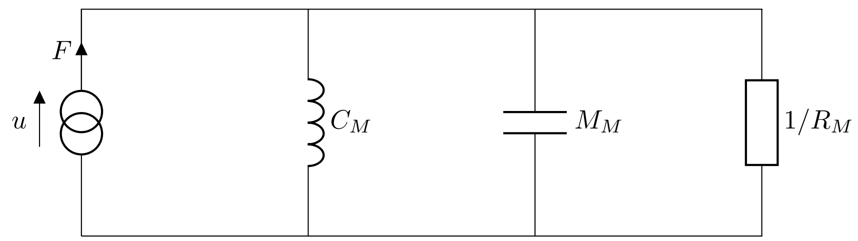


Figure 4.6: Mechanical mobility analogue.

Note that the mechanical system in figure 4.4 is excited externally by an applied force F. According to the mobility analogy $F \to i$ and so our equivalent circuit is driven by an ideal current source.

The force across each mechanical component is now represented by the current flowing through the equivalent electrical component. According to Kirchhoff's current law we have that,

$$F - F_R - F_C - F_M = 0 \tag{4.27}$$

where F_R, F_C and F_M are the force drops across (current flows through) the damper, spring and mass elements. Equation 4.27 is again a statement of Newton's second law.

From the equivalent circuit in figure 4.6 we can determine the total electrical impedance (which is analogous to the mechanical mobility according to the mobility analogy) as,

$$Z_E = \left(\frac{1}{j\omega C} + \frac{1}{\left(\frac{1}{j\omega M}\right)} + \frac{1}{\left(\frac{1}{R}\right)} \right)^{-1} = Y_M = \frac{1}{Z_M}. \tag{4.28}$$

The above simplifies to,

$$Z_E = \left(\frac{1}{j\omega C} + j\omega M + R \right)^{-1} = Y_M = \frac{1}{Z_M}. \tag{4.29}$$

from which the mechanical impedance can be identified as,

$$Z_M \leftrightarrow \frac{1}{Z_E} = j\omega M + \frac{1}{j\omega C} + R. \tag{4.30}$$

Hence, the mobility analogy yields the same mechanical impedance as the impedance analogy, and so the equivalent electrical circuit in figure 4.6 is also entirely analogous to the mechanical system shown in figure 4.4.

4.7 Taking the Dual: Impedance vs. Mobility

The relation between impedance and mobility analogies is a deep one. They are the dual of one another. To take the dual of an equivalent circuit we just have to remember a few key rules:

1) Current source $\leftrightarrow$ voltage source (and vice versa)

2) Capacitor $\leftrightarrow$ inductor (and vice versa)

3) Resistor $\leftrightarrow$ conductor (1/resistor) (and vice versa)

4) Series $\leftrightarrow$ parallel (and vice versa)

4.8 A Closer Look at Mechanical Impedance

Using our equivalent impedance and mobility circuits above we have obtained the mechanical impedance of the mass spring system in figure 4.4,

$$Z_M = R + j\omega M + \frac{1}{j\omega C}. \tag{4.31}$$

Note that the mechanical impedance is frequency dependent and complex. It is convenient rewrite the imaginary part as a single term as so,

$$Z_M = R + j\omega M - \frac{j}{\omega C} \tag{4.32}$$

$$Z_M = R + j\left(\omega M - \frac{1}{\omega C}\right). \tag{4.33}$$

In doing so we have also separated the frequency independent and dependant parts.

The real part of the mechanical impedance is called the resistance, this is not a function of frequency, this kind of damping effects all frequencies equally. The resistance represents all *energy dissipation* in the model; in this case energy is dissipated, via friction, as heat. The *imaginary* part of mechanical impedance is known as the reactance. This is a frequency dependant function that depends on the mass of the driver and the compliance of the suspension. The imaginary part of mechanical impedance represents *energy storage*, i.e. reactive energy is not dissipated.

Looking at the impedance in equation 4.33 we can see there is a contribution due to the mass which increases with frequency ($j\omega M$). This linear increase of the impedance with frequencies implies that for every doubling of frequency (an octave) the mechanical impedance doubles; this is equivalent to saying that due to the mass the impedance increases at a rate of +6dB / octave. Conversely, the compliant term ($\frac{1}{j\omega C}$), halves for every doubling of frequency; therefore, a compliant impedance causes a decrease in mechanical impedance at a rate of -6dB / octave.

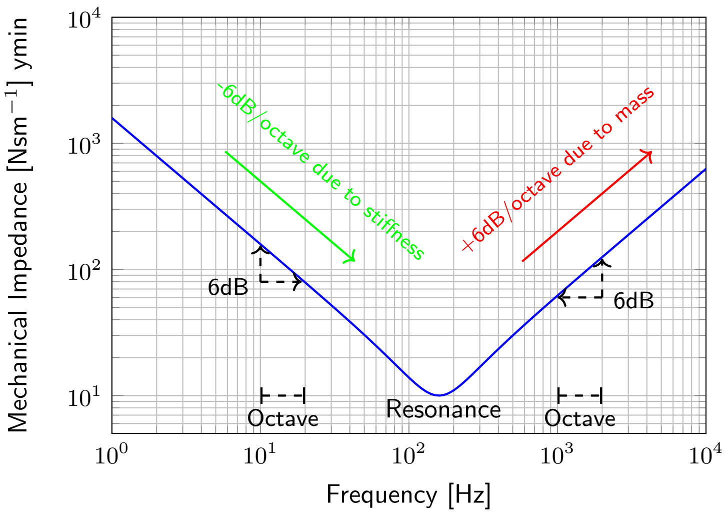


Figure 4.7: Mechanical impedance of a mass spring system (M=0.01 kg, C=1×10-4m/N and R=10 Nsm^{-1})

In Figure 4.7, the value of the impedance at resonance is 1 Nsm^{-1}. At resonance, the reactive motion due to the mass and the reactive motion due to the spring are equal but with opposite phase, thus the reactive part of the impedance is 0 and the impedance takes on its minimum value, i.e. that of the resistance alone. For the reactance to be zero, from 4.33,

$$\left(\omega_c M - \frac{1}{\omega_c C}\right) = 0. \tag{4.34}$$

This leads us to an expression for the resonant frequency,

$$\omega_c = \sqrt{\frac{1}{MC}}. \tag{4.35}$$

Recall from the definition of mechanical impedance that the velocity response due to a force input is the force divided by the mechanical impedance,

$$u = \frac{F}{Z_M} = \frac{F}{R + j\omega M + \frac{1}{j\omega C}}. \tag{4.36}$$

Figure 4.8 shows the velocity response to a 1 N sinusoidal input. The velocity is clearly greatest at resonance, with a +6dB/Oct rise in the stiffness region and a -6dB/Oct fall in the mass region. This is the typical velocity response shape we would expect to see for a loudspeaker. The challenge in loudspeaker design is to turn this resonant response into a flat frequency response at our ears.

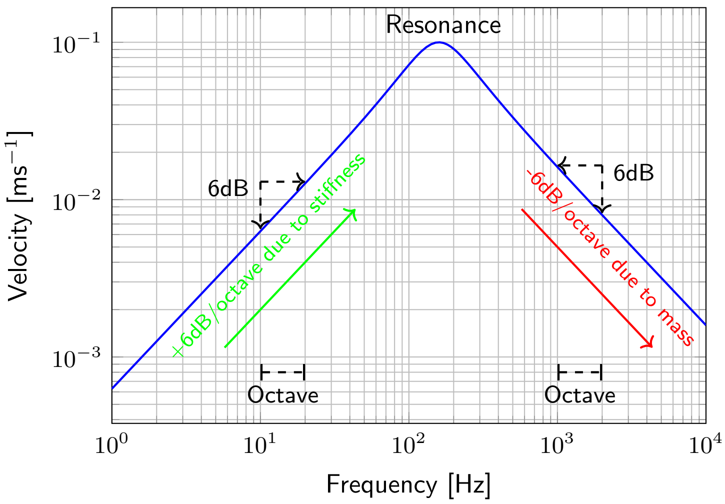


Figure 4.8: Velocity response impedance of a mass spring system (M=0.01 kg, C=1×10-4m/N and R=10 Nsm^{-1})

4.9 Q-factor

The 'peakyness' or sharpness of a resonant response is an important characteristic of vibrating system. We can characterise the sharpness of a resonance using the so called Q-factor, defined as so,

$$Q \triangleq \frac{\omega_c}{\Delta\omega} = \frac{\omega_c}{\omega_2 - \omega_1} \tag{4.37}$$

where ω_c is the resonance frequency of the system, and $\Delta\omega$ is the half power bandwidth (i.e. the spacing between the upper and lower frequencies ω_2 and ω_1 where the output power of the system is half that of the maximum value.)

It is possible to reformulate the Q-factor in terms of the mass, compliance and damping of a system. This form will be particularly useful when we start to look at loudspeaker systems.

To derive this alternate form we begin by recalling the resonant frequency as a function of system mass and compliance,

$$\omega_c = \sqrt{\frac{1}{MC}}. \tag{4.38}$$

Next we must derive an appropriate expression for the full half power bandwidth $\Delta\omega$.

The complex power P of a vibration system is given by,

$$P = u^* F \tag{4.39}$$

where u^* is the complex conjugate of u. Note that we are interested in the *real* power. This is the power that actually does work on the system.

$$\Re(P) = \Re(u^* F) \tag{4.40}$$

Now we can substitute the force F for the product of velocity and impedance,

$$\Re(P) = \Re(u^* u Z). \tag{4.41}$$

Using the identity $z^*z = |z|^2$, the above becomes

$$\Re(P) = |u|^2\Re(Z) \tag{4.42}$$

which leaves us with

$$\Re(P) = |u|^2 R. \tag{4.43}$$

Equation 4.43 states that the real power is proportional to the velocity squared and the damping of the system.

Clearly, the maximum achievable power is that when the velocity is a maximum. Noting that at maximum velocity, the reactance is 0 and the impedance is purely resistive, we have that,

$$P_{max} = |u_{max}|^2 Z = |u_{max}|^2 R. \tag{4.44}$$

The half maximum power can then be written as,

$$\frac{P_{max}}{2} = \frac{|u_{max}|^2}{2} R = \left|\frac{u_{max}}{\sqrt{2}}\right|^2 R. \tag{4.45}$$

We are interested in when the real power $\Re(P)$ is equal to the half maximum power,

$$\Re(P) = \frac{P_{max}}{2} \rightarrow |u|^2 R = \frac{|u_{max}|^2}{2} R \tag{4.46}$$

which is equivalent to,

$$|u| = \left|\frac{u_{max}}{\sqrt{2}}\right|. \tag{4.47}$$

Noting that $|u|\frac{F}{Z}$, the above may be rewritten in the form,

$$\left|\frac{F}{Z}\right| = \left|\frac{F}{R\sqrt{2}}\right| \rightarrow |Z| = R\sqrt{2}. \tag{4.48}$$

Squaring both sides of the above we get,

$$R^2 + X^2 = 2R^2 \tag{4.49}$$

where $|Z|^2 = R^2 + X^2$. This equation has two solutions, $X = \pm R$. Substituting in for the reactance X and doing some minor rearrangement,

$$X = R \qquad\qquad X = -R \tag{4.50}$$

$$\omega_1 M - \frac{1}{\omega_1 C} = R \qquad\qquad \omega_2 M - \frac{1}{\omega_2 C} = -R \tag{4.51}$$

$$\omega_1^2 M - \omega_1 R = \frac{1}{C} \qquad\qquad \omega_2^2 M + \omega_2 R = \frac{1}{C} \tag{4.52}$$

we arrive at an equation for the full half power bandwidth $\Delta\omega = \omega_2 - \omega_1$,

$$\omega_1^2 M - \omega_1 R = \omega_2^2 M + \omega_2 R \rightarrow \omega_2 - \omega_1 = \frac{R}{M}. \tag{4.53}$$

Substituting equation 4.53 and 4.38 into the definition of Q-factor we can derive alternate forms of the Q-factor,

$$Q = \frac{\sqrt{\frac{1}{MC}}}{\frac{R}{M}} = \frac{M}{R}\sqrt{\frac{1}{MC}} = \frac{M\omega_c}{R} = \frac{1}{R}\sqrt{\frac{M}{C}}. \tag{4.54}$$

Of most interest to us is the Q-factor expression,

$$Q = \frac{1}{R}\sqrt{\frac{M}{C}}. \tag{4.55}$$

Lets consider some limiting cases and check equation 4.55 makes sense. What happens as the damping tends to infinity? The Q tends to 0! This makes sense. How about if we set the damping to 0? The Q becomes infinite! This also makes sense.

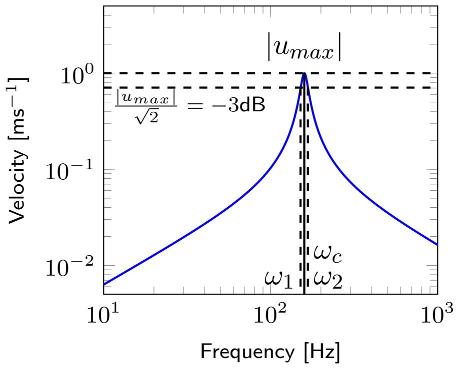


Figure 4.9: High Q-factor response

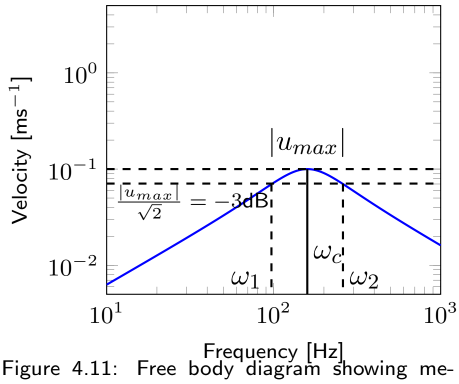


Figure 4.10: Low Q-factor response

4.10 Mass on a Spring the Long Way: Transient Analysis

Having derived the mechanical impedance of a mass-spring system through the use of equivalent circuits and the impedance/mobility analogy, we will now consider a more conventional analysis .

The first step is solving a mechanics problem (such as the motion of a mass-spring system) is to identify all of the forces acting on the body of interest. This is typically done by drawing a free body diagram of the problem. A free body diagram of a loudspeaker cone is shown in figure 4.11. Three forces are identified; the external force applied by the voice coil F_{ext}, the elastic reactionary (/restoring) force provided by the driver's suspension F_k, and its associated resistive frictional force F_R.

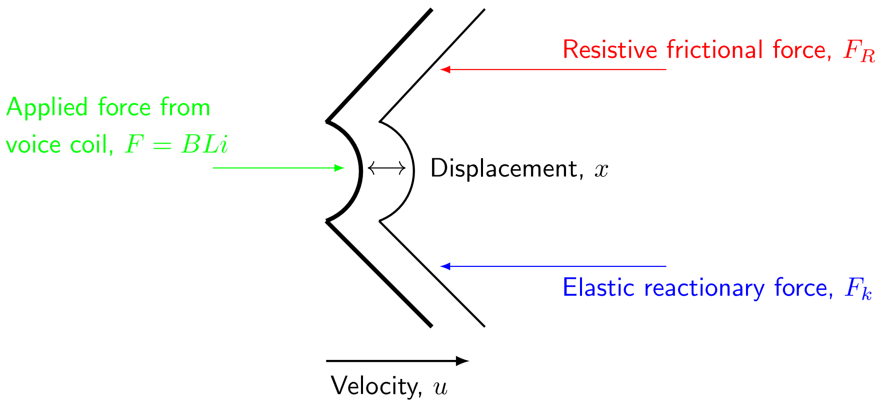


Figure 4.11: Free body diagram showing mechanical forces acting on loudspeaker diaphragm

According to Newton's 2nd law, the total force applied to body is equal to its mass time its acceleration,

$$\sum F = Ma. \tag{4.56}$$

This total force acting on the mass is apparent from Figure 4.11, the resistive and reactive forces are in the opposite direction to the applied motion. The actual direction of the motion is arbitrary, the resistive and elastic elements act to oppose this motion and are therefore negative. Consequently we have that,

$$F_{ext} - F_k - F_R = Ma. \tag{4.57}$$

Substituting the elastic and friction forces for that of a simple spring-dashpot, we can now write our equation of motion,

$$F_{ext} = Ma + \frac{1}{C}x + Ru. \tag{4.58}$$

It is helpful to write equation 4.58 this in terms of a single kinematic variable. In this case we will write each factor in terms of displacement (x). In doing so we get a 2nd order in homogenous differential equation,

$$F_{ext} = M\ddot{x} + \frac{1}{C}x + R\dot{x}. \tag{4.59}$$

Note that there are three main ways of expressing derivatives. Newton and Leibniz both independently developed a theory of calculus; Leibniz's notation is the more familiar but Newton's (and Lagrange's) is the more compact. Equation 4.59 is often referred to as inhomogeneous equation of motion. The homogenous form is given by,

Table 4.1: Different ways of expressing derivatives

Notation	**Displacement**	**Velocity**	**Acceleration**
Newton's	x	$\dot{x}$	$\ddot{x}$
Lagrange's	x	x'	x''
Leibniz's	x	$\frac{dx}{dt}$	$\frac{d^2x}{dt^2}$

$$0 = M\ddot{x} + \frac{1}{C}x + R\dot{x} \tag{4.60}$$

where $F_{ext} = 0$, i.e. the system is allowed to respond freely as it chooses. This is often called the transient equation of motion.

A general solution to the equation of motion as represented by equation 4.59 is the sum of complementary function and particular integral,

$$x = x_{cf} + x_p. \tag{4.61}$$

The complementary function corresponds to the transient solution of the equation of motion, obtained from the homogenous form of the equation of motion. The particular integral corresponds to the steady state solution obtained directly from the inhomogeneous form.

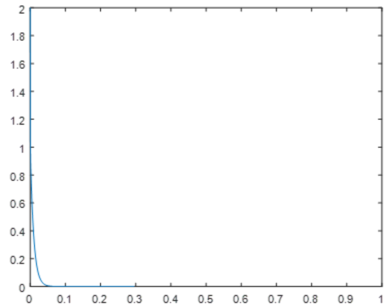


Figure 4.12: Over damped transient response

Starting with the homogenous equation, let us assume a solution of the form $x = Ae^{\gamma t}$,

$$0 = M\gamma^2 Ae^{\gamma t} + \frac{1}{C}Ae^{\gamma t} + R\gamma Ae^{\gamma t} = \left(M\gamma^2 + \frac{1}{C} + R\gamma\right)Ae^{\gamma t} \tag{4.62}$$

If we assume that the above solution is valid for all time, then the value of the exponential term does not matter, and so we can remove it,

$$0 = M\gamma^2 + \frac{1}{C} + R\gamma. \tag{4.63}$$

This is a quadratic equation in γ, i.e. it has two roots! We can find them using the quadratic formula,

$$\gamma = \frac{-R \pm \sqrt{R^2 - 4\frac{M}{C}}}{2M}. \tag{4.64}$$

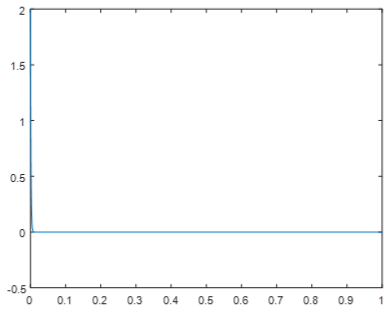


Figure 4.13: Critically damped transient response

Note that our differential equation is linear, and so obeys the principle of superposition; the general solution is the linear combination of both solutions.

$$x_{cf} = Ae^{\gamma_1 t} + Be^{\gamma_2 t} \tag{4.65}$$

where AB and B are constants that depend on the initial conditions. Substituting in for the quadratic roots we have that,

$$x_{cf} = Ae^{\frac{-R+\sqrt{R^2-4\frac{M}{C}}}{2M}t} + Be^{\frac{-R-\sqrt{R^2-4\frac{M}{C}}}{2M}t} \tag{4.66}$$

The above expression can be factored into a more convenient form,

$$x_{cf} = e^{\frac{-R}{2M}t}\left(Ae^{\frac{\sqrt{R^2-4\frac{M}{C}}}{2M}t} + Be^{\frac{\sqrt{R^2-4\frac{M}{C}}}{2M}t}\right) \tag{4.67}$$

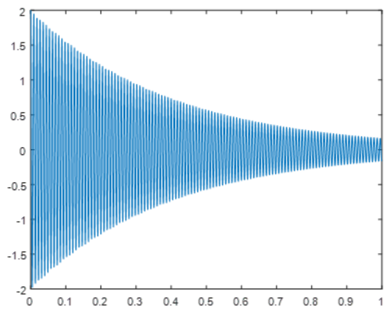


Figure 4.14: Under damped transient response

The first exponential term acts as decay term, whose rate depends on the amount of damping R and the mass M. The second bracketed term corresponds to a potential oscillation term; depending on whether the argument of the square root is positive or negative this term will either yield another exponential decay, or an oscillation.

There are 3 distinct scenarios regarding the above.

1) $4\frac{M}{C} < R^2$ - the square root is positive so we have real roots and no oscillation (over damped).

2) $4\frac{M}{C} = R^2$ - the square root is 0 so we have the decay term only and no oscillation (critically damped).

3) $4\frac{M}{C} > R^2$ - the square root is negative so we have complex roots ($\sqrt{-N} = \sqrt{(-1)N} = \sqrt{-1}\sqrt{N} = j\sqrt{N}$) and oscillation (under damped).

These 3 conditions can be expressed in terms of the Q-factor as so,

1) $Q < \frac{1}{2}$ - over damped

2) $Q = \frac{1}{2}$ - critically damped

3) $Q > \frac{1}{2}$ - under damped

To get the steady state solution we consider a periodic applied force $F = F_0 e^{j\omega t}$

$$F = F_0 e^{j\omega t} = M\ddot{x} + \frac{1}{C}x + R\dot{x}. \tag{4.68}$$

Since we are dealing with a linear system, we know that the response will also be periodic with the same frequency, $x = x_0 e^{j\omega t}$.

$$F_0 e^{j\omega t} = \left(-\omega^2 M + \frac{1}{C} + j\omega R\right) x_0 e^{j\omega t} \tag{4.69}$$

The steady state solution is then given by,

$$x_s = \frac{F}{-\omega^2 M + \frac{1}{C} + j\omega R} = \frac{F}{j\omega\left(R + j\left[\omega M - \frac{1}{\omega C}\right]\right)} \tag{4.70}$$

The complete solution is now obtained by adding together the transient and steady state solutions,

$$x = e^{\frac{-R}{2M}t}\left(Ae^{\frac{\sqrt{R^2 - 4M\frac{1}{C}}}{2M}t} + Be^{\frac{\sqrt{R^2 - 4M\frac{1}{C}}}{2M}t}\right) + \frac{F}{\omega^2 M + \frac{1}{C} + j\omega R}. \tag{4.71}$$

Note that as $t \to \infty$ the transient part of the solution will tend to 0. We will focus primarily on the steady state part in our loudspeaker design.

Finally, note that by taking the derivative of the stead state response derived above,

$$u = \frac{dx}{dt} = \frac{d}{dt}\frac{F}{j\omega\left(R + j\left[\omega M - \frac{1}{\omega C}\right]\right)} = \frac{F}{\left(R + j\left[\omega M - \frac{1}{\omega C}\right]\right)} \tag{4.72}$$

we arrive at exactly the same velocity response we obtained using our equivalent circuit approach.

5 Acoustic Domain

So far we have covered the electrical and mechanical domains. The final domain of interest is the acoustic domain. The acoustic domain encompasses all the radiative aspects of loudspeaker design. This includes not only the propagation of sound to a receiver, but also the effect of air loading on the mechanical domain (i.e. radiation impedance).

5.1 Acoustic Impedance and Mobility

The acoustic domain, like the electrical and mechanical, has a notion of impedance. In fact, the acoustic domain has 3 different notions of impedance! Let us begin by first recalling the definitions of electrical and mechanical impedance,

$$Z_E = \frac{V}{i} \tag{5.1}$$

$$Z_M = \frac{F}{u}. \tag{5.2}$$

In the acoustic domain our state variables are pressure and either particle or volume velocity. The former leads to a definition of specific acoustic impedance; the ratio of acoustic pressure to particle velocity, i.e. the velocity at which the particles of air in an acoustic wave move.

$$Z_{As} = \frac{p}{u}. \tag{5.3}$$

The specific acoustic impedance describes the impedance a wave sees as it propagates through a medium. It described the relationship between the pressure and particle velocity, which is *specific* for that medium and that wave type (e.g. plane waves and spherical waves have a different specific impedance).

The use of volume velocity leads to a definition of the (usual) acoustic impedance; the ratio of acoustic pressure to volume velocity,

$$Z_A = \frac{p}{U}. \tag{5.4}$$

Another commonly used variant of the acoustic impedance is that of the radiation impedance, defined as the specific acoustic impedance multiplied by the area of the radiating surface.

$$Z_{Ar} = S\frac{p}{u} \tag{5.5}$$

It basically describes how efficiently a surface can radiate.

The (usual) acoustic impedance Z_A is the most common form, and so it is useful to define its reciprocal value, the acoustic mobility,

$$Y_A = \frac{U}{p}. \tag{5.6}$$

5.1.1 Volume Velocity

Volume velocity is defined as the product of the normal component of surface velocity u, with the area of that surface S. For a uniformly vibrating surface, for example a sphere or a disk, it is given quite simply as velocity times total area.

$$U = Su \tag{5.7}$$

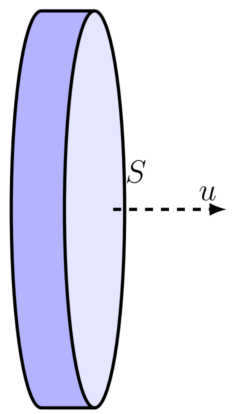


Figure 5.1: Volume velocity of a rigid disk.

For more complex vibrating geometries the differential volume velocity,

$$dU = \hat{n} \cdot u dS \tag{5.8}$$

where $\hat{n}$ is the unit vector normal to the differential surface dS, can be integrated across the structures surface to yield a total volume velocity.

From equation 5.7 we can see that the units of volume velocity are m^3/s. We can see why its called a volume velocity. We have meters cubed, i.e. volume, per second, i.e. velocity. Also, unlike particle or surface velocity, volume velocity isn't a vector, it's a scalar (i.e. it doesn't have a direction, only a magnitude).

5.1.2 Acoustic vs. Mechanical Impedance

As in the electrical and mechanical domain, there are 3 main acoustic elements for which we can define an acoustic impedance (and mobility). Like the mechanical domain, these correspond to mass, spring and damper like elements. We will cover these elements shortly, but first it is important to demonstrate the relation between mechanical and acoustic impedance.

Noting that pressure is defined as the force per area, $p = F/S$, or

$$F = Sp, \tag{5.9}$$

the mechanical impedance can be rewritten as,

$$Z_M = \frac{pS}{u} \tag{5.10}$$

which can be further rearranged as so,

$$p = \frac{uZ_M}{S}. \tag{5.11}$$

Substituting the above into the definition of acoustic impedance yields,

$$Z_A = \frac{p}{U} = \frac{p}{uS} = \frac{\left(\frac{uZ_M}{S}\right)}{uS} = \frac{Z_M}{S^2}. \tag{5.12}$$

Equation 5.12 states that the acoustic impedance Z_A is related to the mechanical impedance Z_M by surface area squared.

5.2 Acoustic Mass

A lump of air that moves as a single unit will behave as if it were a mass-like element. Consequently, it obeys Newton's second law,

$$F = M_M \frac{du}{dt} \tag{5.13}$$

where M_M is the mechanical mass (in kg) of the lump of air, u is its velocity, and F is the applied force.

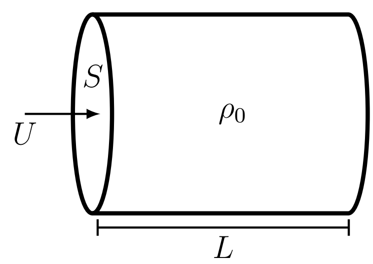


Figure 5.2: Acoustic mass element.

When dealing with acoustic elements it is useful to express their behaviour in terms of acoustic variables, i.e. pressure and volume velocity. Dividing both sides of the above by S and then multiplying the top and bottom of the right hand side by S we obtain,

$$\frac{F}{S} = \frac{M_M}{S}\frac{du}{dt} \rightarrow \frac{F}{S} = \frac{M_M}{S}\frac{1}{S}\frac{d(uS)}{dt}. \tag{5.14}$$

Noting that F/S is the definition of pressure, and that volume velocity is given by $uS = U$, the above is equivalent to,

$$p = M_A \frac{d(U)}{dt} \tag{5.15}$$

where we have introduced the definition of 'acoustic mass', $M_A = M_M/S^2$. As in the mechanical domain, if we assume a periodic excitation (i.e. $p = p_0 e^{j\omega t}$), the derivative can be evaluated quite easily and we obtain,

$$p = j\omega M_A U. \tag{5.16}$$

From equation 5.16 we can readily obtain the acoustic impedance of a (lump) mass of air,

$$Z_A = \frac{p}{U} = j\omega M_A. \tag{5.17}$$

Similarly, the mobility can be obtained as,

$$Y_A = \frac{U}{p} = \frac{1}{j\omega M_A}. \tag{5.18}$$

Note that the impedance (and mobility) of an acoustics mass is identical in form to that of a mechanical mass. The only difference is that we use the acoustic mass M_A, as opposed to the mechanical mass M_M. As we have previously shown, these are related by a factor of S^2.

Before moving onto an acoustic compliance lets look a little more closely at the acoustic mass. Substituting the mechanical mass for the element's length L, surface area S and density ρ_0, we have that,

$$M_A = \frac{M_M}{S^2} = \frac{LS\rho_0}{S^2} = \frac{L\rho_0}{S}. \tag{5.19}$$

Perhaps surprisingly equation 5.19 indicates that the acoustic mass of a lump of air is *inversely proportional* to its surface area! As you increase the surface area (and so the size of the element) the acoustic mass decreases. This is a little counter intuitive, but important to remember.

It is important to note that the above treatment for an acoustic mass assumes that no waves are able to propagate within the mass itself. This is assumption is fundamental to the application of the lumper parameter approach we are considering.

5.3 Acoustic Compliance

Now onto the compliance, or 'spring' element. An acoustic compliance can be thought of as a cavity or box of air, with an opening over which a fluctuating pressure can act. According to our lumped parameter assumption, we can not have any wave propagation within the element. This means that the pressure

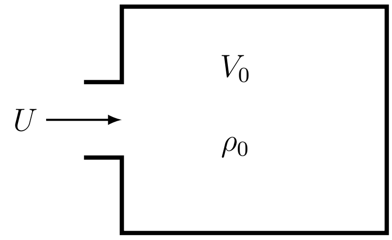


Figure 5.3: Acoustic compliance element.

is the same across the entire cavity, and remains so as an external pressure is applied.

The equation that governs the compression of a volume of air by some net force is given as so,

$$F = \frac{1}{C_M}\int u dt \tag{5.20}$$

where C_M is the mechanical compliance of the volume. The above equation can be converted into acoustic units following the same approach as the mass element,

$$\frac{F}{S} = \frac{1}{C_M S}\frac{1}{S}\int (uS)dt. \tag{5.21}$$

The above is entirely equivalent to,

$$p = \frac{1}{C_A}\int U dt \tag{5.22}$$

where we have introduced the definition of 'acoustic compliance', $C_A = C_M S^2$. Assuming a periodic excitation, the above yields the acoustic impedance,

$$Z_A = \frac{p}{U} = \frac{1}{j\omega C_A} \tag{5.23}$$

and the acoustic mobility,

$$Y_A = \frac{U}{p} = j\omega C_A. \tag{5.24}$$

Note that the impedance (and mobility) of an acoustics compliance is identical in form to that of a mechanical spring.

The acoustic compliance of a volume can be related to the volume of the cavity V and the properties of the enclosed gas as so,

$$C_A = \frac{V}{\rho_0 c^2} \tag{5.25}$$

where ρ_0 is the density of air and c is the speed of sound.

5.4 Acoustic Damping

The final acoustic element to consider is that of an obstruction, or an acoustic damping material. Any structure which dissipates the energy of acoustic particles, for example a fine mesh, or a fibrous material like foam, can act as an acoustic damper. The acoustic resistance through such a material is proportional to the particle velocity (i.e. fast particles face a greater resistance). This is similar to the viscous damping law in our mechanical system.

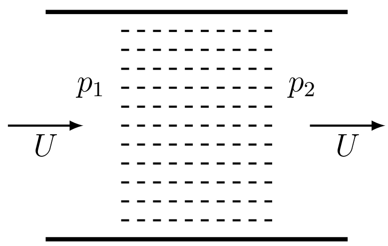


Figure 5.4: Acoustic resistance element.

The governing equation for the resistance faced by an acoustic wave is given by,

$$F = R_M u. \tag{5.26}$$

Conversion into acoustical units yields,

$$p = R_A U \tag{5.27}$$

where we have introduced the 'acoustic resistance', $R_A = \frac{R_M}{S^2}$. The acoustic impedance of an obstruction is then given by,

$$Z_A = \frac{p}{U} = R_A \tag{5.28}$$

and its acoustic mobility by,

$$Y_A = \frac{U}{p} = \frac{1}{R_A}. \tag{5.29}$$

Note that the impedance (and mobility) of an acoustics obstruction/damper is identical in form to that of a mechanical resistance, and is independent of frequency.

5.5 Acoustic Circuits

Now that we have derived the acoustic impedance and mobility of mass, spring, and damping like elements, we can consider the formulation of equivalent circuits that describe acoustical systems. We will introduce the equivalent acoustic circuit by way of an example.

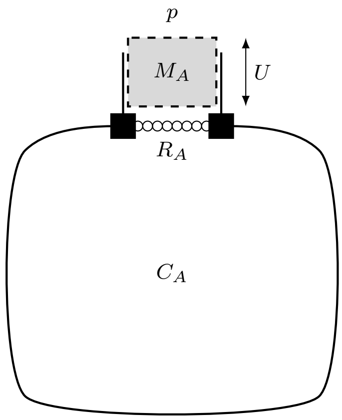


Figure 5.5: Acoustic Helmholtz resonator.

Shown in figure 5.5 is a Helmholtz resonator. It consists of an acoustic cavity to which a short duct is attached with a fine mesh within. The lump of air that sits within the short duct behaves as a mass like element. This acoustic mass sits on top the acoustic compliance provided by the cavity. The fine mesh housed within the duct acts like an acoustic obstruction/damper, opposing the motion of the acoustic mass as it 'bounces' on top of the cavity.

The acoustic system that is the Helmholtz resonator can be modelled using an equivalent circuit, much like our previous mass on a spring. To form an equivalent circuit we must first chose an analogy to follow. The analogy dictates whether we relate pressure to voltage and volume velocity to current or visa vera. As in the mechanical domain, we have two choices, the so called impedance and mobility analogies.

The impedance analogy retains an equivalence between the notion of impedance, such that,

$$Z_A \sim Z_E. \tag{5.30}$$

To achieve this we must have that,

$$p \leftrightarrow V \qquad \text{(Drop)} \tag{5.31}$$

$$U \leftrightarrow i \qquad \text{(Flow)}. \tag{5.32}$$

The mobility analogy instead draws an equivalence between electrical impedance and acoustic mobility, such that,

$$Y_A = \frac{1}{Z_A} \sim Z_E. \tag{5.33}$$

To achieve this we must have that,

$$U \leftrightarrow V \qquad \text{(Drop)} \tag{5.34}$$

$$p \leftrightarrow i \qquad \text{(Flow)}. \tag{5.35}$$

Which analogy we choose to adopted will depend on the problem at hand, but also on personal preference. Like the mechanical domain, the impedance analogy has the advantage of retaining an equivalence between the impedance in each domain, whilst the mobility analogy has the advantage of retaining the topology of the problem.

5.5.1 Impedance Analogy

From inspection we can see that the impedance of an acoustic mass is identical in form to the impedance of an inductor, $M_A \to L_E$. Similarly, we can see that the impedance of an acoustic compliance is identical in form to the impedance of a capacitor, $C_A \to C_E$, and that an acoustic damper is equivalent to an electrical resistor, $R_A \to R_E$.

According to the impedance analogy, the volume velocity of an acoustic element is equivalent to the current flowing through the equivalent electrical element, and the pressure is equivalent to its voltage drop. We can see from figure 5.5 that as the mass element oscillates with volume velocity U, the obstruction and cavity both see the same volume velocity. In an equivalent circuit this corresponds to each component seeing the same current. For each component to have the same current the circuit must be in series, as in figure 5.6.

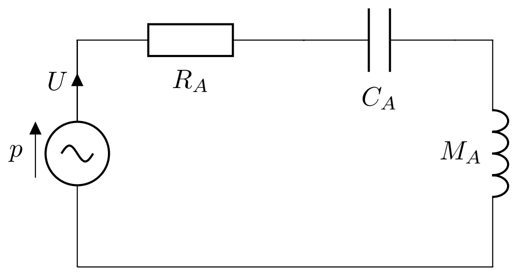


Figure 5.6: Acoustic impedance analogue.

Note that the acoustic system in figure 5.5 is excited externally by an applied pressure p. According to the impedance analogy $p \to V$ and so our equivalent circuit is driven by an ideal voltage source.

From figure 5.6 we can determine the total acoustic impedance of the Helmholtz resonator by finding the equivalent electrical impedance. For a series circuit this is simply,

$$Z_E = j\omega M_A + \frac{1}{j\omega C_A} + R_A = Z_A. \tag{5.36}$$

From the above it is clear that determining the volume velocity of the acoustic mass element is equivalent to find the current in the equivalent circuit,

$$i = \frac{V}{Z_E} \leftarrow U = \frac{p}{j\omega M_A + \frac{1}{j\omega C_A} + R_A}. \tag{5.37}$$

This is exactly what we would get is we followed a more conventional analysis.

In summary, the equivalent electrical circuit in figure 5.6 is entirely analogous to the acoustic system shown in figure 5.5.

5.5.2 Mobility Analogy

To form an equivalent circuit according to the mobility analogy, each acoustic element is represented by an equivalent electrical component whose impedance is of the same form as the acoustic element's mobility. Consequently, the acoustic mass element is represented by an electrical capacitor (with capacitance M_A), the acoustic compliance by an inductor (with inductance C_A), and the acoustic damping by a resistor (whose resistance is one over the damping coefficient R_A).

According to the mobility analogy, the volume velocity of an acoustic element is equivalent to the voltage drop across its equivalent electrical component. Similarly, the pressure is equivalent to the current flowing through the equivalent electrical component. Again, note that the mass, cavity and obstruction all share the same volume velocity; in an equivalent circuit this corresponds to each component seeing the same voltage. For each component to have the same voltage drop the circuit must be in parallel, as in figure 5.7.

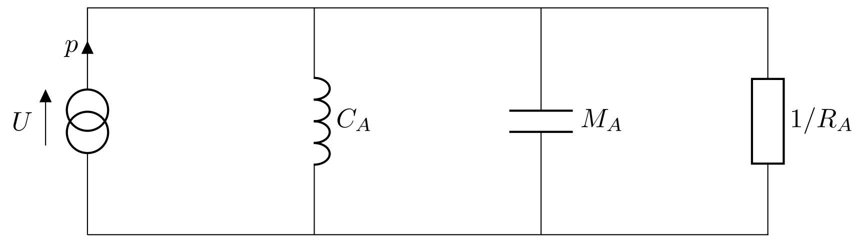


Figure 5.7: Acoustic mobility analogue.

Note that the acoustic system in figure 5.5 is excited externally by an applied pressure p. According to the mobility analogy $p \to I$ and so our equivalent circuit is driven by an ideal current source.

From figure 5.7 we can determine the total acoustic impedance of the Helmholtz resonator by first finding the equivalent electrical impedance. For a parallel circuit this is simply,

$$Z_E = \left(\frac{1}{j\omega C_A} + \frac{1}{\frac{1}{j\omega M_A}} + \frac{1}{\frac{1}{R_A}} \right)^{-1}. \tag{5.38}$$

Recalling the mobility analogue relation $Z_A = 1/Z_E$, the acoustic impedance is obtained as,

$$Z_A = j\omega M_A + \frac{1}{j\omega C_A}. \tag{5.39}$$

This is exactly what we obtain using the impedance analogy!

In summary, the equivalent electrical circuit in figure 5.7 is *also* entirely analogous to the acoustic system shown in figure 5.5.

6 Coupling Domains

So far we have introduced the mechanical and acoustic domains, and shown that by adopting a particular analogy (impedance or mobility based), equivalent circuits can be employed to model physical systems. In this chapter we will introduce the theory of ideal transformers and show how it can be used to couple together our electrical, mechanical and acoustical circuits. This will enable us to form a single equivalent circuit encompassing the three domains of a loudspeakers operation.

6.1 Transducers

The act of converting one form of energy (or domain) to another is called transduction. The electro-dynamic loudspeaker employed two forms of transduction; from the electrical domain to the mechanical domain (so called electro-mechanic transduction), and from the mechanical domain to the acoustic domain (so called mechano-acoustic transduction). In order to couple the electrical, mechanical and acoustical domains it is first necessary to introduce the transduction equations that relate their respective state variables.

6.1.1 Electro-mechanical

There are two main types of electro-mechanical transduction. One is called electro-dynamic and the other is called electro-static. We will consider only electro-dynamic transduction, as this is used by most conventional loudspeakers.

An electro-dynamic transducer is based on the interaction between a static magnetic field and a dynamic electrical field that surrounds a conductor (i.e. the voice coil). Any conductor carrying a current in a magnetic field is subject to a Lorentz force. It is this Lorentz force that drives the motion of a loudspeaker diaphragm.

We will cover the electo-magnetic aspects of loudspeakers in more detail later on.

The total Lorentz force generated by the voice coil is related to the strength of the applied magnetic field B (i.e. a stronger magnet will give use a greater force), the length of the voice coil wire L, and the current i running through it,

$$F = Bli. \tag{6.1}$$

Electro-dynamic transduction is a two way phenomena. When a conductor moves in a magnetic field, a voltage is generated across its length. This voltage is proportional to the field strength B, the coil length L, and the conductor's velocity u,

$$V = Blu. \tag{6.2}$$

The voltage V is often called the back EMF (electro-motive force). It is an induced voltage that opposes the flow of current. It has the effect of reducing the overall current.

$$i = i_{applied} - i_{bEMF} \tag{6.3}$$

6.1.2 Mechano-acoustical

Transduction from the mechanical to acoustic domain is straight forward and the governing equations have already been introduced (see equations 5.7 and 5.9),

$$F = Sp \tag{6.4}$$

$$U = Su. \tag{6.5}$$

The above equations relate mechanical force F and velocity u, to acoustic pressure p and volume velocity U.

Equations 6.1, 6.2, 6.4, and 6.5 describe the electro-acoustical transduction of an electro-dynamic loudspeaker. It happens that their mathematical form is identical to those of an ideal transformer.

6.2 Ideal Transformers

Transformers are passive electrical devices which transfers electrical energy between two or more circuits (see figure 6.1 for a diagrammatic illustration). A widespread application is to step up or (down) voltages and currents. For example, if you want to send a current over a long distance, you encounter far fewer losses when you use a transformer to step down the current and step up the voltage. Then on the other side you just step it the other way and recover the current.

The fundamental operation of a transformer is described by Faraday's law of induction. A time varying current flowing around a primary coil, through which we have a magnetic core, induces a time varying magnetic field in the core. Around another part of the core we have a second coil. The time varying magnetic field then induces an Electro-Motive Force (EMF) across its length.

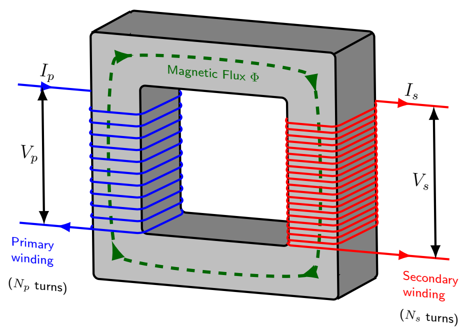


Figure 6.1: Ideal transformer

We are interested in the characteristics of ideal transformers only. This means that a) there is a perfect coupling between the coil and the magnetic flux, and b) that the coils have 0 resistance and 0 inductance. This amounts to the assumption that the power supplied to the transformer, is equal the power supplied by the transformer. Recalling the definition of electrical power,

$$P = Vi \tag{6.6}$$

an ideal transformer satisfies the following equation,

$$P_{in} = P_{out} \longrightarrow V_p i_p = V_s i_s \tag{6.7}$$

where V_p and V_s are the voltages across the primary and secondary side windings, respectively. Similarly, i_p and i_s are the currents flowing through the primary and secondary side windings, respectively. The above can be rearranged as so,

$$\frac{V_p}{V_s} = \frac{i_s}{i_p}. \tag{6.8}$$

Equation 6.8 states that if you step up the voltage across the transformer, then the current is reduced proportionally. This is the conservation of energy.

It can be shown that the turns ratio α of the transformer (i.e. the ratio of the number of turns on the primary and secondary windings) is equal to the ratio of primary and secondary side voltages,

$$\frac{V_p}{V_s} = \frac{N_p}{N_s} = \alpha. \tag{6.9}$$

From equation 6.8 we also have that the change in current is determined by the reciprocal of this same ratio,

$$\frac{i_p}{i_s} = \frac{1}{\alpha}. \tag{6.10}$$

6.3 Equivalent Circuit Transformers

It happens that the equations governing the transformation of voltage and current across an ideal transformer are identical in form to those of our electro-mechanical and mechano-acoustical transduction.

From equation 6.9 we have that the primary side transformer voltage V_p is related to the secondary side voltage V_s through the turns ratio α. Similarly, from equation 6.2 we have that the voltage across the voice coil of a loudspeaker V is related to the diaphragm velocity u through the force factor Bl.

$$V_p = \alpha V_s \longleftrightarrow V = Blu \tag{6.11}$$

From equation 6.10 we have that the primary side transformer current i_p is related to the secondary side current i_s through the reciprocal turns ratio $1/\alpha$. Similarly, from equation 6.1 we have that the current through the voice coil of a loudspeaker i is related to the force F applied to the diaphragm through the reciprocal force factor $1/Bl$.

$$i_p = \frac{1}{\alpha} i_s \longleftrightarrow i = \frac{1}{Bl} F \tag{6.12}$$

From the above it is clear that an ideal transformer whose turn ratio α is equal to the force factor Bl will implement exactly the electro-mechanical transduction of a loudspeaker.

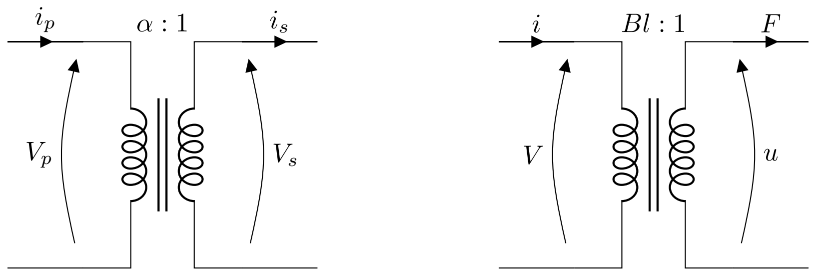


Figure 6.2: Ideal transformer vs. electro-mechanical coupling.

The electro-mechanical transformer shown in figure 6.2 provides a means of coupling an equivalent mechanical circuit to the electrical domain. Note that on the output of the electro-mechanical transformer we have that force is equivalent to current, and velocity is equivalent to voltage. This is the *mobility analogy*. To incorporate the dynamics of mechanical domain we can simply substitute in our mobility based equivalent circuit for a mass spring system (see figure 4.6). The coupled electro-mechanical system is now represented by the equivalent circuit in figure 6.3. Note that in figure **??** we have introduced two new electrical components, a resistor and inductor. The purpose of these components is to model the resistive and inductive properties of the voice coil.

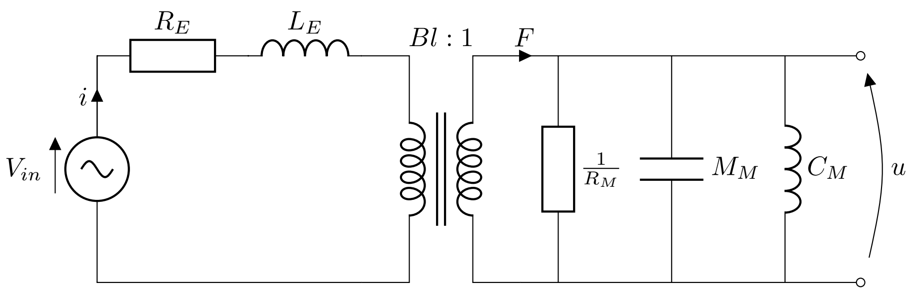


Figure 6.3: Ideal transformer coupling between electrical and mechanical domains.

Now let us consider the mechno-acoustic transduction. From equation 6.9 we have that the primary side transformer voltage V_p is related to the secondary side voltage V_s through the turns ratio α. Similarly, from equation 6.5 we have that the velocity of the loudspeaker diaphragm u is related to the acoustic volume velocity U through the reciprocal of surface area $1/S_D$.

$$V_p = \alpha V_s \longleftrightarrow u = \frac{1}{S}U \tag{6.13}$$

From equation 6.10 we have that the primary side transformer current i_p is related to the secondary side current i_s through the reciprocal turns ratio $1/\alpha$. Similarly, from equation 6.4 we have that the force applied to a loudspeaker F is related to the acoustic pressure p through the surface area S_D.

$$i_p = \frac{1}{\alpha} i_s \longleftrightarrow F = Sp \tag{6.14}$$

From the above it is clear that an ideal transformer whose turn ratio α is equal to the reciprocal of surface area $1/S_D$ will implement exactly the mechano-acoustical transduction of a loudspeaker.

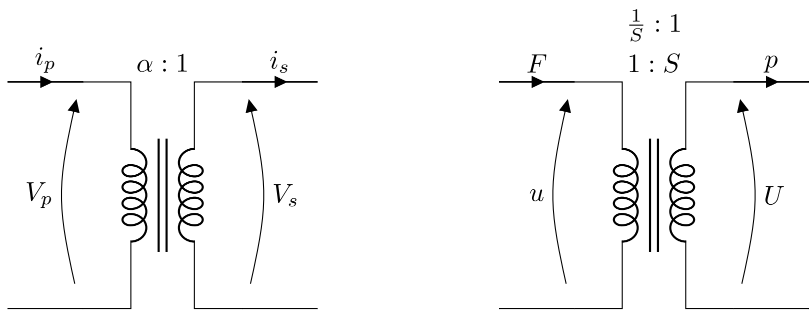


Figure 6.4: Ideal transformer vs. mechano-acoustical coupling.

The mechano-acoustical transformer shown in figure 6.4 provides a means of coupling equivalent mechanical and acoustical circuits. Note that on the output of the mechano-acoustical transformer we have that pressure is equivalent to current, and volume velocity is equivalent to voltage. This is again the *mobility analogy*. To incorporate the dynamics of acoustic domain we simply place the appropriate mobility based equivalent circuit onto the right hand side of the mechano-acoustical transformer. The coupled mechano-acoustical system is now represented by the equivalent circuit in figure 6.5.

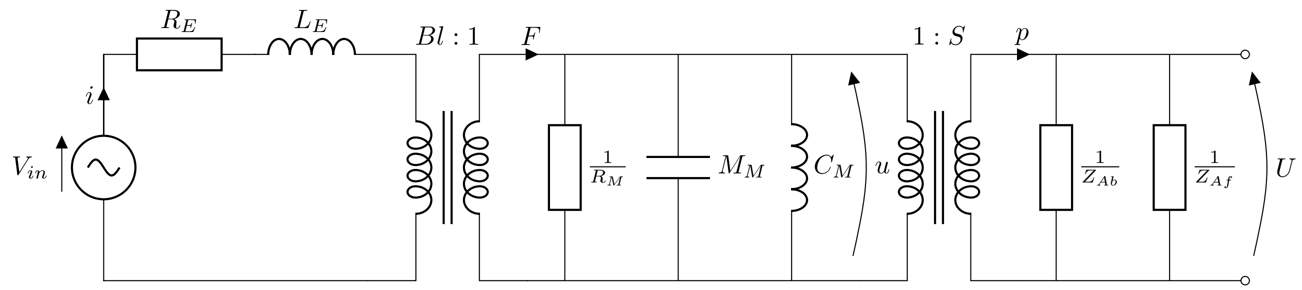


Figure 6.5: Ideal transformer coupling between mechanical and acoustical domains.

Note that in figure 6.5 we have introduced two arbitrary acoustical elements, represented by the impedances Z_{Af} and Z_{Ab}. The purpose of these elements are to account for the acoustic loading on the front and rear of the loudspeaker driver. The two elements have been arranged in parallel as the mechano-acoustical transformer yields a mobility analogue on its output. The air load in front and behind of the driver experience the same volume velocity (albeit 180 degrees out of phase) as they are both in direct contact with the driver. According to the mobility analogy $U \to V$, and so to achieve the same voltage the two elements must be in parallel.

The exact form of Z_{Af} and Z_{Ab} will be dictated by the type of cabinet (if any) the loudspeaker driver is housed in. Later we will consider the sealed and vented cabinet in particular.

6.4 Removing Transformers

Now that we have successfully coupled our three domains we are almost in a position to start analysing our equivalent circuit. Unfortunately, the electro-mechanical and mechano-acoustical transformers complicate this analysis. We are therefore interested in removing the transformers to get a more simple circuit that we can analyse using our AC circuit theory.

The removal of a transformer from an electrical circuit involves moving all components to one particular side (primary or secondary) by applying an appropriate scaling to voltage, current and impedance. The act of moving a component across a transformer and scaling it accordingly is called 'referring the impedance to the primary (or secondary) side'. The form of the scaling will depend on whether the components are moved from the primary to secondary side, or visa versa.

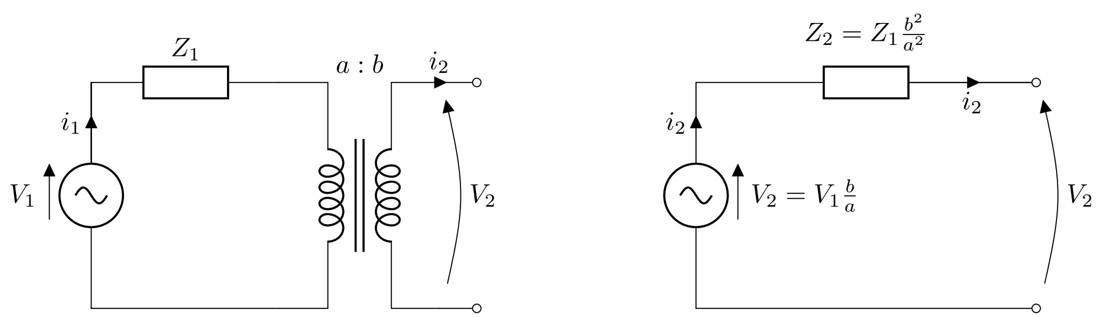


Figure 6.6: Moving electrical components across a transformer - left to right

Consider the simple transformer circuit in the left hand side of figure 6.6. On the primary side we have an ideal voltage source V_1 in series with an arbitrary impedance Z_1, through which a current i_1 flows. A transformer with a turns ratio

a/b connects a second circuit, whose voltage and current are V_2 and i_2, respectively. We are interested in moving the voltage source and impedance element across to the secondary side. This involves finding the equivalent voltage, current and impedance that generates the same output as the transformer's secondary side.

We have already shown that the current through a transformer scales according to the turns ratio (see equation 6.10),

$$i_2 = i_1 \frac{a}{b}. \tag{6.15}$$

Similarly, the voltage scales according to the reciprocal of the turns ratio (see equation 6.9),

$$V_2 = V_1 \frac{b}{a}. \tag{6.16}$$

From the above it is clear that a primary side impedance must be scaled according to the reciprocal turns ratio squared,

$$Z_2 = \frac{V_2}{i_2} = \frac{V_1 \frac{b}{a}}{i_1 \frac{a}{b}} = Z_1 \left(\frac{b}{a}\right)^2. \tag{6.17}$$

The scaled voltage, current and impedance above will yield an output identical to that of the transformer, i.e. the two circuits in figure 6.6 are entirely equivalent.

Now consider the simple transformer circuit in the left hand side of figure 6.7. On the primary side we have a voltage source V_1 connected directly to a transformer of turns ratio a/b, through which a current i_1 flows. On the transformer's output a second circuit is connected containing a series impedance element Z_2. The voltage across and current through the secondary winding are V_2 and i_2, respectively. We are interested in moving the impedance element across to the primary side.

By rearranging equations 6.15 and 6.16 we have that,

$$i_1 = i_2 \frac{b}{a} \tag{6.18}$$

and

$$V_1 = V_2 \frac{a}{b}. \tag{6.19}$$

From the above it is clear that the impedance must be scaled according to the turns ratio squared,

$$Z_1 = \frac{V_1}{i_1} = \frac{V_2 \frac{a}{b}}{i_2 \frac{b}{a}} = Z_1 \left(\frac{a}{b}\right)^2. \tag{6.20}$$

The scaled impedance Z_1 presents the same load to the voltage source as the transformer did with Z_2 on its output.

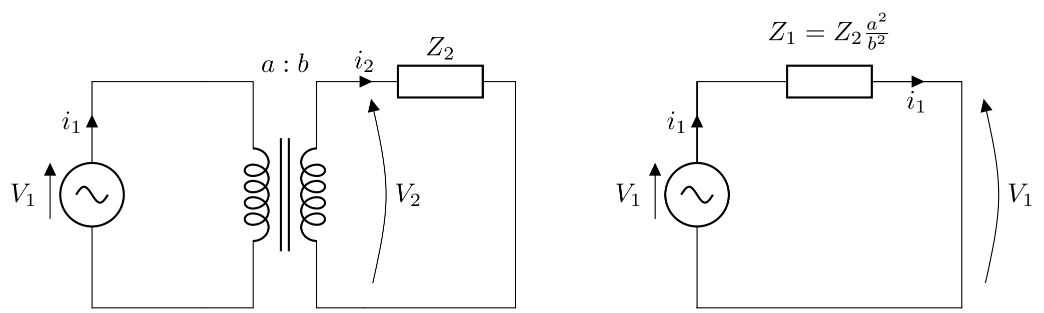


Figure 6.7: Moving electrical components across a transformer - left to right

In summary, the rules for referring an impedance to the primary side (moving components to the left) are:

- Scale impedance by the turns ratio squared.
- Scale voltages by the turns ratio.
- Scale currents by the reciprocal of the turns ratio.

For referring an impedance to the secondary side (moving components to the right):

- Scale impedance by the reciprocal of the turns ratio squared.
- Scale voltages by the reciprocal of the turns ratio.
- Scale currents by the turns ratio.

Using these rules we are now able to remove the transformers from our equivalent loudspeaker circuit in figure 6.5.

6.5 Complete Equivalent Circuit

When removing the transformers from our equivalent circuit we can either move the components to the left (into the electrical domain) or to the right (into the acoustic domain). For now we are interested in the determining the acoustic response of our loudspeaker and so we will consider the latter. We will consider the movement of components into the electrical domain later when we consider the electrical impedance of our loudspeaker.

Removal of the transformers from figure 6.5 is a two step procedure. First we must transfer all electrical domain components (and sources) into the mechanical domain through the electro-mechanical transformer with turns ratio Bl. Next all the mechanical domain components (including the newly transferred electrical components) are transferred into the acoustic domain through the mechano-acoustic transformer with turns ratio $1/S$. The resulting circuit is shown in figure 6.8.

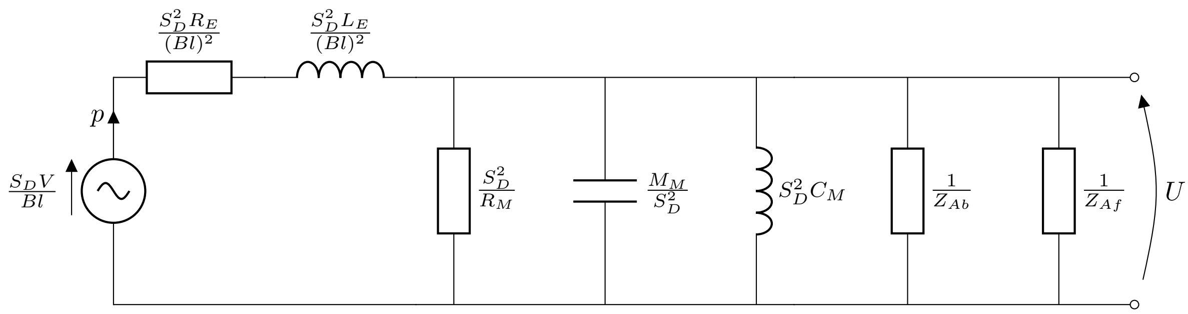


Figure 6.8: Equivalent circuit with transformers removed.

Note that the electrical resistance R_E and inductance L_E have been scaled by $S_D^2/(Bl)^2$, i.e. the combined squared reciprocal turns ratio of both transformers. Similarly, the voltage source V has been scaled by the reciprocal turns ratio of both transformers. The mechanical impedances have each been scaled by S_D^2, i.e. the squared reciprocal turns ratio of the mechano-acoustic transformer. Recall that the impedance of the capacitor (representing mechanical mass) is $Z_C = 1/j\omega M_M$, and so its scaled capacitance is given by M_M/S_D^2.

Figure 6.8 represents a complete equivalent circuit model of our loudspeaker in the acoustic domain. Note that it follows the mobility analogy where, $V \to U$

and $i \to p$. A minor simplification can be made by recalling our definitions of acoustic mass, compliance and damping. The subscript AD is used to denote the *driver's* properties once transferred into the *acoustic domain*.

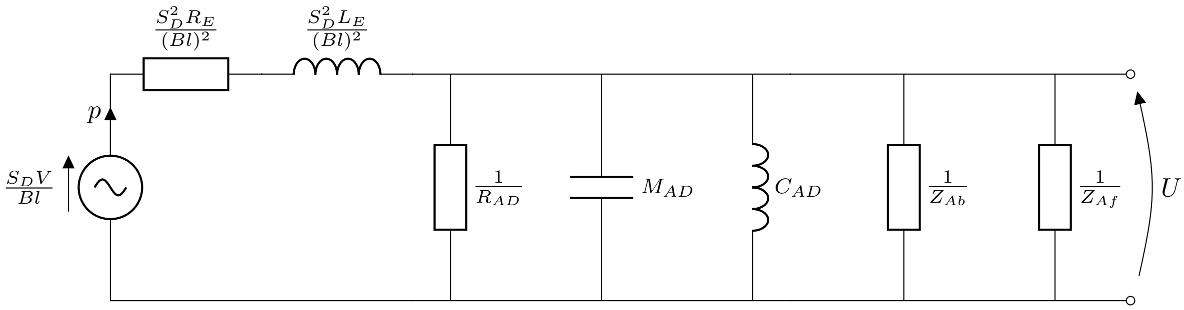


Figure 6.9: Equivalent circuit with transformers removed with mechanical impedance replaced by acoustic impedance.

Note that our equivalent circuit is composed mostly of parallel elements, with the exception of the electrical resistance and inductance. Although this is amenable to our AC circuit analysis, further simplifications are available. We would much prefer to analyse a series circuit... We know that a strictly parallel circuit can be converted into a series circuit by taking its dual. However, our equivalent circuit isn't strictly parallel! The series electrical impedance ruin this for us. What we want to do is replace these two series elements with some equivalent parallel impedance. This sounds exactly like a good for Norton's theorem.

6.5.1 Application of Norton's Theorem

Using Norton's theorem it is possible to replace a voltage source in series with a known impedance, with an equivalent Norton current source in parallel with an appropriate Norton impedance, as in figure 6.10.

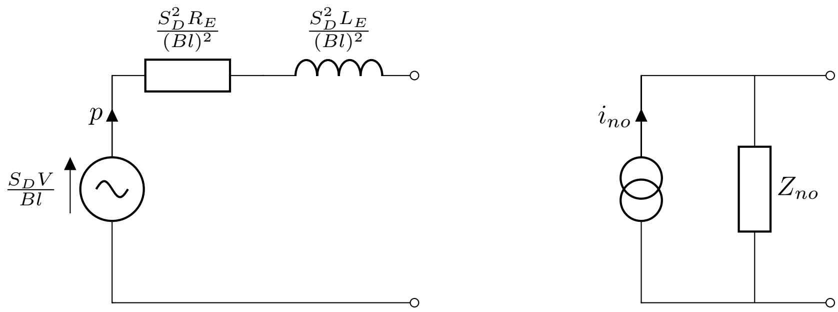


Figure 6.10: Equivalent Norton circuit for electrical components in loudspeaker model.

The procedure for determining the Norton current source and impedance has already been outlined and demonstrated on the simple circuit in figure 3.16. To determine the Norton impedance Z_{no} the voltage source is short circuited and the impedance is found across the circuit's output terminals. From figure 6.10 the Norton impedance is given by,

$$Z_{no} = \frac{R_E S_D^2}{(Bl)^2} + \frac{j\omega L_E S_D^2}{(Bl)^2}. \tag{6.21}$$

To obtain the Norton current i_{no} the circuit's output is short circuited and its current found. From figure 6.10 the Norton current is given by,

$$i_{no} = \frac{V}{Z_{no}} = \frac{\frac{V S_D}{Bl}}{\frac{R_E S_D^2}{(Bl)^2} + \frac{j\omega L_E S_D^2}{(Bl)^2}}. \tag{6.22}$$

Multiplying top and bottom by $(Bl)^2/S_D$ then yields,

$$i_{no} = \frac{VBl}{S_D(R_E + j\omega L_E)}. \tag{6.23}$$

Now, according to Norton's theorem, the circuits of figure 6.9 and 6.11 are entirely equivalent. Figure 6.11 however, is strictly parallel!

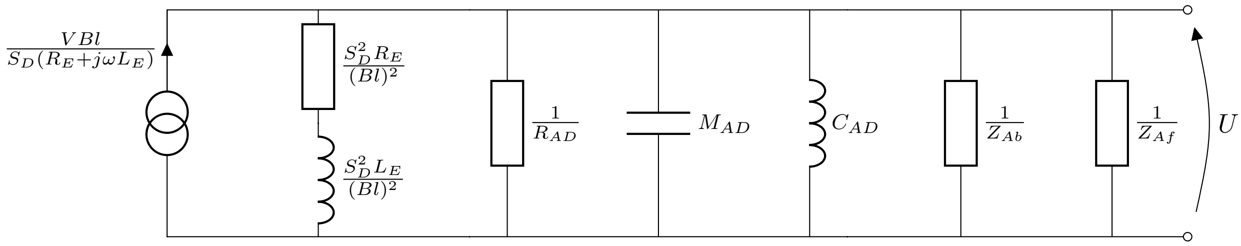


Figure 6.11: Equivalent circuit with transformers removed and Norton's theorem applied.

6.5.2 Taking the Dual

Now that we have an entirely parallel equivalent circuit, we take take its dual to obtain a simple series circuit. Recall the rules for taking the dual of an equivalent circuit (i.e. converting from a mobility to impedance analogy):

1) Current source $\leftrightarrow$ voltage source (and vice versa)

2) Capacitor $\leftrightarrow$ inductor (and vice versa)

3) Resistor $\leftrightarrow$ conductor (1/resistor) (and vice versa)

4) Series $\leftrightarrow$ parallel (and vice versa)

Applying the above set of rules to circuit in figure 6.11 leads to an impedance based formulation of our equivalent circuit loudspeaker model, as in figure 6.12.

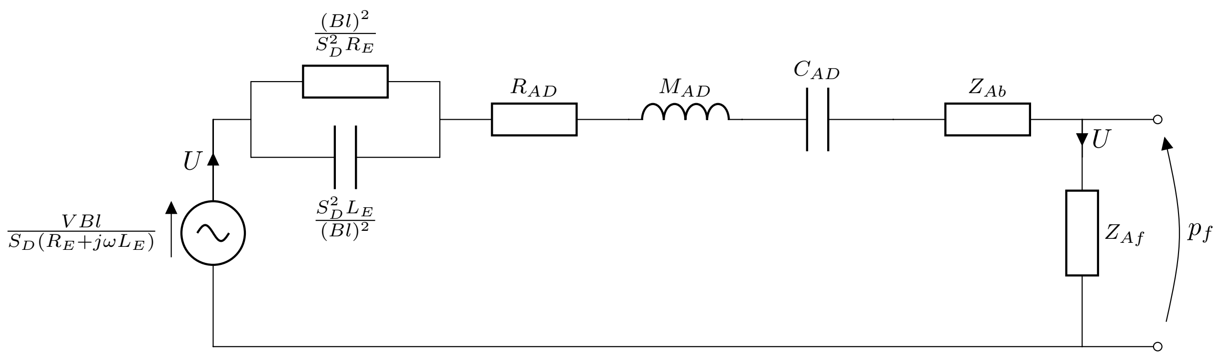


Figure 6.12: Equivalent circuit with transformers removed and Norton's theorem applied.

Figure 6.12 represents a complete low frequency equivalent circuit model of a loudspeaker driver loaded by two arbitrary acoustic impedances. The volume velocity procedure by the speaker is equivalent to the circuit's current. The form of the acoustic loading Z_{Af} and Z_{Ab} will dictate whether the loudspeaker is modelled in an infinite baffle or a sealed cabinet. To model a vented cabinet some small modifications must be made to the circuit to account for the dynamics of the vent.

Before we can applying an appropriate acoustic loading however, we must consider the radiation of sound from a loudspeaker driver. Doing so will enable us derive the acoustic loading when radiating into free space (i.e. for a loudspeaker in an infinite baffle). The effect of cabinetry can then be introduced by deriving an appropriate acoustic back load Z_{Ab}.

7 Sound Radiation

In the previous chapter we developed an equivalent circuit model that describes the low frequency (i.e. lumped parameter) behaviour of a moving coil loudspeaker. Thus far we have not looked at how sound is *radiated* from loudspeakers, and so we are unable to complete our equivalent circuit (we still need to assign the acoustic loading). In this chapter we will consider the radiation of sound from simple 'idealised' sources, namely, the monopole, dipole and rigid piston. Based on the radiation of a rigid piston we will derive a lumped parameter approximation for the acoustic loading of the air surrounding a driver. This will allow us to model the radiation of loudspeaker systems in the subsequent chapter, including infinite baffle, sealed and vented cabinets.

7.1 Monopole

The acoustic monopole is the simplest acoustic source we have. Conceptually, a monopole can be thought of as a rigid sphere whose radius is made to expand and contract periodically. This pulsation causes a compression and rarefaction in the surrounding medium which then propagates as an acoustic wave. In the limit that the radius of the monopole sphere tends to 0 we arrive at a so called 'point source'. The key feature of a monopole, or point source, is that it radiates sound in all directions equally, i.e. the surrounding pressure field depends only on distance. Although the monopole is an idealised source, at low frequencies most sound sources tend behave approximately like monopoles.

One of the useful things about monopoles is that they can be combined in various arrangements to model more complex, and thus realistic, acoustic sources. We will see this shortly when we look at dipole and piston sources.

The equation governing the acoustic radiation from a monopole is,

$$p(r,t) = \frac{j\rho_0 cka^2 u}{r} e^{j(\omega t - kr)} \tag{7.1}$$

where ρ_0 is the density of air, c is the speed of sound, $k = \omega/c$ is wave number, a is the sphere radius, u is the surface velocity and r is the distance. The derivation of equation 7.1 is covered in your *Principle of Acoustics* notes.

The radiated pressure surrounding a monopole is shown in figure 7.1. An important feature of this pressure is its dependence on frequency. Note that the radiate pressure is a function of wave number k, which itself is a function of frequency ω. Consequently, for a fixed surface velocity u, the radiated pressure p from a monopole increases linearly with frequency.

Recalling the surface area of a sphere, $V = 4\pi a^2$, the volume velocity of a

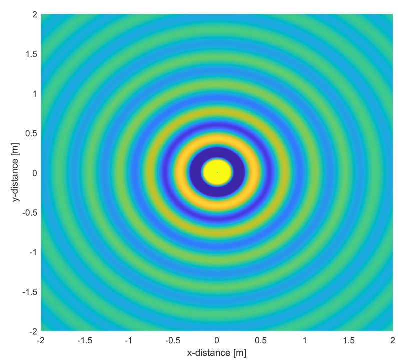


Figure 7.1: Directivity of a monopole

monopole with radius a and surface velocity u is given by,

$$U = u4\pi a^2. \tag{7.2}$$

Substitution into equation 7.1 then yields,

$$p(r,t) = \frac{j\rho_0 ckU}{4\pi r} e^{j(\omega t - kr)}. \tag{7.3}$$

Suppose a monopole were placed against an infinite baffle. The rear radiated sound would be reflected by the baffle and redirected forwards. If the monopole is place directly against the infinite baffle there would be no time delay in this reflection, and so the radiate pressure would be in-phase and we get a doubling in the sound pressure. Consequently, the radiation of an infinitely baffled monopole is given by,

$$p(r,t) = \frac{2A}{r} e^{-jkr} \tag{7.4}$$

where,

$$A = \frac{j\rho_0 ckU}{4\pi} e^{j\omega t}. \tag{7.5}$$

Equation 7.4 provides a reasonable model for the low frequency radiation of a loudspeaker in an infinite baffle. At higher frequencies, where the drivers dimensions become comparable to the radiating wave length, this model breaks down and a more sophisticated approach is required.

7.2 Dipole

The acoustic dipole is the next simplest source we have. It is made up of two monopoles separated by a small distance d radiating $90°$ out of phase. Its surrounding pressure field can thus be written as,

$$p(r,\theta,t) = \frac{A}{r} e^{j(-kr)} + (-)\frac{A}{r + \Delta r(\theta)} e^{j(-k(r+\Delta r(\theta)))} \tag{7.6}$$

where r is the distance from leading monopole to the receiver position, and Δr is the small additional distance (positive or negative) travelled from the second monopole, as in figure 7.2.

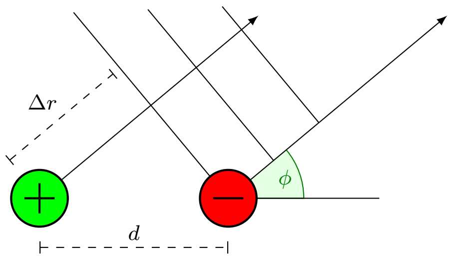


Figure 7.2: Geometrical configuration of a dipole

From figure 7.2 the additional distance Δr can be rewritten in terms of the monopole spacing, and the polar angle θ,

$$\Delta r = d\cos\theta. \tag{7.7}$$

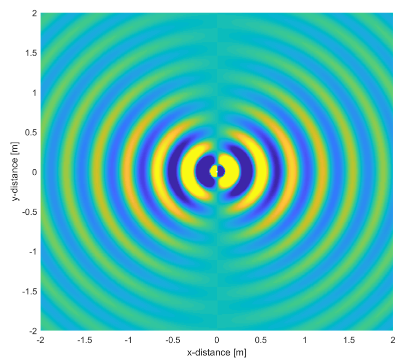


Figure 7.3: Directivity of a dipole.

Substitution into equation 7.6 then yields,

$$p(r,\theta,t) = \frac{A}{r} e^{-jkr} - \frac{A}{r + d\cos\theta} e^{-jk(r+d\cos\theta)} \tag{7.8}$$

If we assume a far field radiation the attention due to distance for the two monopoles will be approximately the same we can rewrite the above as so,

$$p(r,\theta,t) = \frac{A}{r} e^{-jkr}\left(1 - e^{-jkd\cos\theta}\right). \tag{7.9}$$

Note that the first term in equation 7.9 corresponds to that of a standard monopole. The second term accounts for the change in directivity due to the second out of phase monopole. Recalling Euler's formula this directivity term can be rewritten as,

$$1 - e^{-jkd\cos\theta} = 1 - \cos(kd\cos\theta) + j\sin(kd\cos\theta). \tag{7.10}$$

Assuming the $kd << 1$, i.e. the spacing between the monopoles is small and/or we consider low frequencies, the $\cos$ and $\sin$ functions can be replaced by their small argument approximations $\cos x \approx 1$ and $\sin x \approx x$,

$$1 - e^{-jkd\cos\theta} \approx 1 - 1 + jkd\cos\theta = jkd\cos\theta. \tag{7.11}$$

Substituting the above into equation 7.9 yields the dipole equation,

$$p(r, \theta, t) = \frac{A}{r} e^{-jkr} \times jkd\cos\theta. \tag{7.12}$$

The radiated pressure field of a dipole, as per equation 7.12, is shown in figure 7.3.

Equation 7.12 provides a reasonable model for the low frequency radiation of a loudspeaker suspended in free space as in figure 7.4 (so that the front and rear radiated sound can interfere). At higher frequencies, where the drivers dimensions become comparable to the radiating wave length, this model breaks down and a more sophisticated approach is required.

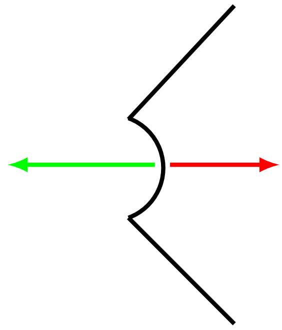


Figure 7.4: Radiation from a loudspeaker in free space.

7.3 Piston

At very low frequencies the monopole and dipole provide reasonable models for the radiation of sound from a loudspeaker. However, once the radiated wavelength becomes comparable to the loudspeaker's size, these simple models breakdown and more complex ones are required.

Assuming that the motion of loudspeaker's driver remains purely pistonic (i.e. no wave motion) we can approximate its radiation using a rigid piston of equal surface area.

A rigid piston in an infinite baffle can be modelled mathematically by arranging a large number of monopoles across the surface of an imaginary piston and summing together their contributions. By taking the limit as the number of monopoles tends to infinity, this summation becomes an integral and we have,

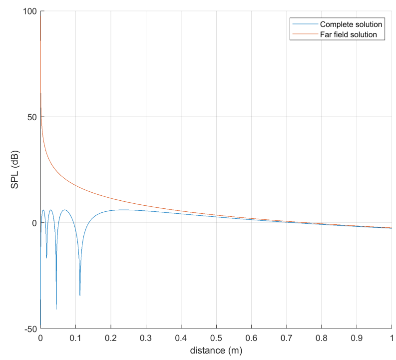


Figure 7.5: Loudspeaker radiation using the far field approximation compared against monopole summation model

$$p(r, \theta, t) = \int_S \frac{A}{r} e^{-jkr} dS = A \int_S \frac{1}{r} e^{-jkr} dS \tag{7.13}$$

where S is the surface area of the piston and $\frac{A}{r}e^{-jkr}$ is the radiation of a single monopole.

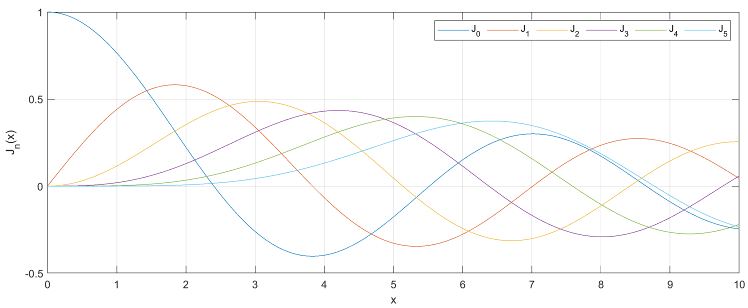


Figure 7.6: Bessel functions of the zeroth to fifth kind

Evaluating the integral in equation 7.13 is not straight forward and so we wont cover it here (you will cover piston radiation in more detail in *Principles of Acoustics*). Assuming far field radiation, equation 7.13 can be solved to yield,

$$p(r, \theta, t) = \frac{j\rho_0 cka^2 u}{2r} e^{j(\omega t - kr)} \left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta} \right] \tag{7.14}$$

where a is the radius of the piston, u is its surface velocity, and $J_1(\quad)$ is a first order Bessel function of the first kind (see figure 7.6). The first group of terms bear a resemblance to the monopole equation, it does not contain any angular dependence. The second term is a directivity factor.

Shown in figure 7.5 are the on-axis ($\theta = 0$) radiated pressures predicted using a monopole summation (with 100 monopoles) and the far field approximation of equation 7.14. From figure 7.5 we can see that the far field approximation does not account for the detailed response at small distances (due to interference from different parts of the driver). In the far field ($r > 1$ m) equation is in excellent agreement with the monopole summation.

Many of the terms in equation 7.14 will already be familiar from the monopole and dipole sources introduced above. The new term not yet introduced is the Bessel function $J_1(\quad)$. Bessel functions are the canonical solutions of Bessel's differential equation. They are defined by their order, which can be specified by any complex number. Integer and half integer orders are the most relevant to us, as they represent solutions to the wave equation in cylindrical and spherical coordinates. We wont go into any of the details about how they arise. We just want to know what they look like.

Shown in figure 7.6 are the Bessel functions of order 0, 1, 2, 3, 4 and 5. In appearance they look quite similar to decaying sinusoids. The directivity factor in equation 7.14 depends not on the Bessel function alone, but on $2J_1(v)/v$, where the argument $v = ka \sin\theta$ depends on the angle θ.

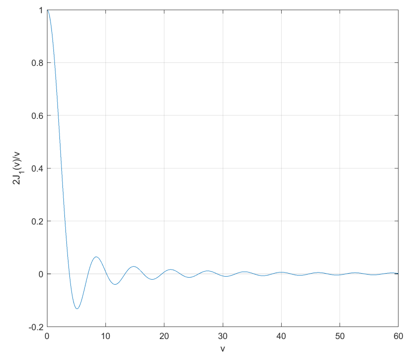


Figure 7.7: Functional dependence of directivity function $2J(v)/v$.

7.3.1 Directivity

Shown in figure 7.8 is the function dependence of the directivity factor,

$$DF(ka, \theta) = \left[\frac{2J_1(ka \sin\theta)}{ka \sin\theta}\right] \tag{7.15}$$

for a radius $a = 0.15$ m, at a frequency of 15000 Hz. Note that the argument of the directivity factor $ka \sin\theta$ varies between 0 and ka for $\theta = 0 \to 90°$, with 0 being its minimum value ($\sin 0° = 0$) and ka being its maximum value ($\sin 90° = 1$). Due to the periodic nature $\sin\theta$, the directivity factor will begin to repeat itself after $90°$. Therefore we need not evaluate the directivity beyond ka. We call this the visible region.

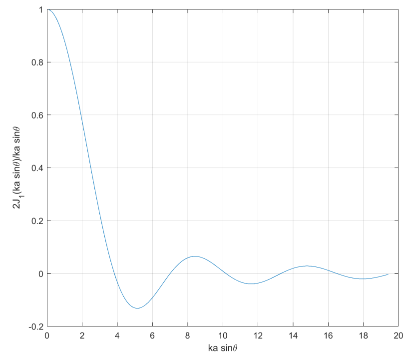


Figure 7.8: Piston radiation at high frequency - $r = 0.15$ m, $f = 15000$ Hz, $ka = 19.4227$.

From figure 7.8 we can see that between $\theta = 0°$ and $90°$ we get a series of peaks and nulls. These correspond to lobes in our directivity function. A value of one indicates complete radiation, a value of 0 indicated no radiation. E.g. at $ka \sin\theta \approx 4$ we get a minimal radiation from our piston. Note that the maximum value of $ka \sin\theta$ corresponds to the value of $ka = 19.4227$.

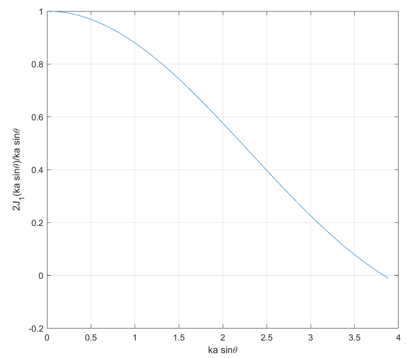


Figure 7.9: Piston radiation at mid frequency - $r = 0.15$ m, $f = 3000$ Hz, $ka = 3.8845$.

Shown in figure 7.9 is the directivity factor for the same piston at a frequency of 3000 Hz. Notice that the maximum argument is now $ka \sin\theta = 3.8845$ (corresponding to $\theta = 90°$). Consequently, the main lobe has been stretched out and we see only 1 peak.

Shown in figure 7.10 is the directivity factor for the same piston at a frequency of 150 Hz. Notice that the maximum argument is now $ka \sin\theta = 0.1942$ (corresponding to $\theta = 90°$). Here, the main lobe has been stretched out even further such that it covers the entire range of θ. Figure 7.10 indicates that at low frequencies the rigid piston radiates equally in all directions (i.e. is omni-directional), much like a monopole.

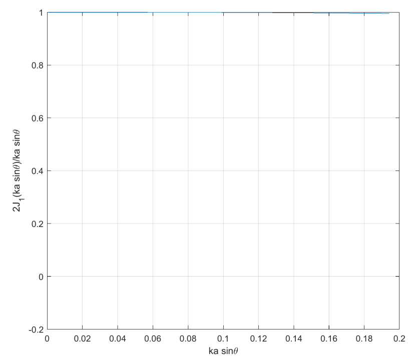


Figure 7.10: Piston radiation at low frequency - $r = 0.15$ m, $f = 150$ Hz, $ka = 0.1942$.

Whilst figures 7.8-7.10 show the directivity as a function of angle, it is more common to plot them in polar form as in figure 7.11. Here we can clearly see a the effect of beaming, and the introduction of side lobes, as frequency is increased.

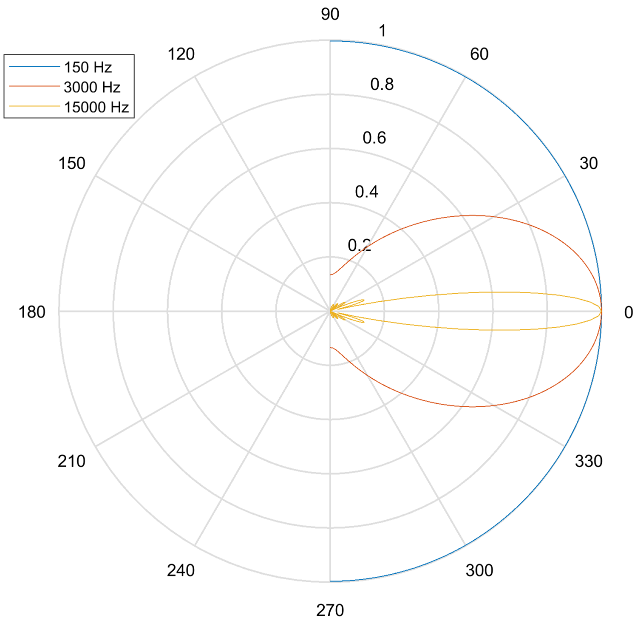


Figure 7.11: Polar response at 150, 3000 and 15000 Hz for a piston with $r = 0.15$ m.

Figures 7.8-7.10 clearly illustrate the importance of ka in characterising the directivity of the piston. The value ka defines the point in the directivity function which corresponds to $90°$, and consequently to point at which the directivity starts repeating. It defines the upper limit of our directivity function. By changing the frequency or piston area we essentially shift this point up an down. For low frequencies or small piston areas, the point ka is well within the main lobe and we have a flat angular response. For high frequencies or large piston areas, the point ka extends well beyond the main lobe, and we have a highly directional response.

Substituting $k = 2\pi f/c = 2\pi/\lambda$, we have that

$$ka = 2\pi \frac{a}{\lambda}. \tag{7.16}$$

Equation 7.16 demonstrates that ka may be interpreted as the ratio of piston radius a to radiating wavelength λ. A value of $ka >> 1$ indicated a greater radius than wavelength and a directional response. A value of $ka << 1$ indicated a greater wavelength than radius and an omni-directional response.

7.4 Radiation Impedance

So we have discussed the *radiation* of acoustic waves due to a rigid piston. We haven't discussed the resistance to motion that this diaphragm will face when oscillating. In a vacuum a piston can oscillate freely; there is no air to impede its motion. When placed in free space (air included), the properties of the surrounding medium (e.g. its inertia) impede the piston's motion. This phenomenon we call acoustic impedance.

As loudspeaker diaphragm oscillates it sees an impedance due to the surrounding air. This needs to be account for in our equivalent circuit model.

The radiation impedance of a rigid piston can be obtained by integrating the specific acoustic impedance (radiated pressure over surface velocity, see equation 5.3), over the radiating surface area,

$$Z_{rad} = \int_S \frac{p}{u} dS. \tag{7.17}$$

Evaluating the above integral is not straightforward and so we wont covered it in detail here. The resulting expression is,

$$Z_{M,rad} = \rho_0 c S \left[J_1(2ka) + jX_1(2ka) \right] \tag{7.18}$$

where $J_1(\quad)$ is a Bessel function of the first kind, and $X_1(\quad)$ is the so called Struve function. Equation 7.18 represents the radiation impedance of the piston in the mechanical domain. To get the acoustic domain equivalent we must divide by surface area squared,

$$Z_{A,rad} = \frac{\rho_0 c}{S} \left[J_1(2ka) + jX_1(2ka) \right]. \tag{7.19}$$

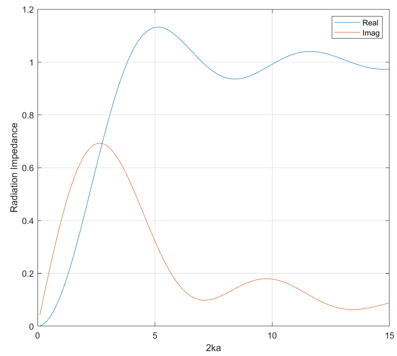


Figure 7.12: Frequency dependence of the radiation impedance

The frequency dependence of equation 7.19 is shown in figure 7.12. Note that because the piston functions are functions of ka, the radiation load varies with both frequency and driver size.

Some broad features of the piston radiation impedance are:

- At high frequencies ($ka > 2$) $J_1(\quad)$ is close to one and $X_1(\quad)$ is small so that the radiation load approximates the plane-wave characteristic value of $\rho_0 c$ scaled by S. So at high frequencies, the radiation load is mostly real and so looks a little like that of an infinite pipe.

- At low frequencies ($ka < 0.25$) the radiation load is very small. But we need a high load to generate a high pressure level. When the radiation load is small, the volume velocity that is shoved through this thing doesn't generate much pressure (from ohms law, $p = UZ$).

Note that we are only interested in modelling the low frequency behaviour of loudspeakers, i.e. for small values of ka. With this in mind we can take a first order approximation of equation 7.19 (this is done by expanding equation 7.19 as a Taylor series, and discarding all but the first term). Doing so yields,

$$Z_{A,rad} \approx \frac{\rho_0 c k^2}{2\pi} + j\omega \frac{8\rho_0}{3\pi^2 a}. \tag{7.20}$$

Equation 7.20 represents a first order approximation of the acoustic load faced by a rigid piston in an infinite baffle. Note that the real part is proportional to ω^2 ($k = \omega/c$), and that the imaginary part is linearly proportional to ω. These are the low frequency trends observed in figure 7.12.

From equation 7.20 we can associate the real and imaginary parts of the radiation impedance with resistive and inertial effects. The resistive term (i.e. the real part) is quadratic in frequency. This terms describes the power dissipated. This is what we want, i.e. to dissipate power as sound energy. The reactive term looks quite similar to the impedance of a mass. It is proportional to frequency and related to density and area. The reactance acts like a lump of air glued to the speaker cone. Its effect is to increase the apparent mass of the driver and in turn lower the resonant frequency of the system.

From the above it is clearly advantageous to have a purely real load; more sound would be dissipated and there would be no change in the resonant frequency of the diaphragm.

7.5 Infinite Duct Acoustics

Before moving onto loudspeaker systems it is useful to consider the acoustics of an infinite duct. Why exactly this is important/useful wont be clear until later when we consider transmission line loudspeakers.

Let us assume that we have an acoustic wave travelling in our infinite duct. If the wave length of sound is much greater than the cross section of the duct we can assume that only plane waves are propagating (no standing waves across the width or height of our duct). Also, because we are dealing with an infinite duct, there will be no reflections of any sort, and so we can assume our wave travels in only one direction. An acoustic plane wave travelling in one direction is described by the equation,

$$p^+ = p_0 e^{j(\omega t - kx)}. \tag{7.21}$$

We are interested in finding the acoustic impedance as seen by this travelling wave.

We begin by recalling Euler's equation which related the spatial gradient of pressure to the time derivative of particle velocity,

$$\frac{dp}{dx} = -\rho_0 \frac{du}{dt}. \tag{7.22}$$

Note that Euler's equation can be interpreted as different form of Newton's 2nd Law $F = ma$. A pressure gradient generates a force, density is related to mass, and the time derivative of particle velocity is acceleration.

Integrating both sides of Euler's equation with respect to time yields the following equation for velocity,

$$u = -\frac{1}{\rho_0} \int \frac{dp^+}{dx} dt. \tag{7.23}$$

Now substituting in our plane wave,

$$u = -\frac{p_0}{\rho_0} \int \frac{d}{dx} e^{j(\omega t - kx)} dt \tag{7.24}$$

evaluating the derivative,

$$u = -\frac{jkp_0}{\rho_0} \int e^{j(\omega t - kx)} dt \tag{7.25}$$

and then the integral,

$$u = -\frac{jkp_0}{j\omega\rho_0} e^{j(\omega t - kx)} \tag{7.26}$$

we arrive at an equation relating particle velocity to pressure,

$$u - \frac{k}{\omega\rho_0} p^+ = -\frac{1}{\rho_0 c} p^+. \tag{7.27}$$

Recall that the ratio of pressure to particle velocity is the specific acoustic impedance,

$$Z_s = \frac{p^+}{u} = \rho_0 c. \tag{7.28}$$

Through the definition of volume velocity $U = uS$, the standard acoustic impedance can be determined as so,

$$Z_A = \frac{p^+}{U} = \frac{\rho_0 c}{S}. \tag{7.29}$$

Equation 7.29 represents the acoustic impedance as seen by a travelling plane wave in an infinite duct. Why is this interesting? It is purely resistive (no imaginary parts!). This means that attaching an infinite pipe to a loudspeaker will have no effect on its resonant frequency (this is controlled by the reactive terms in its impedance). We will use this notion later when we look at transmission line loudspeaker systems.

8 Loudspeaker Systems

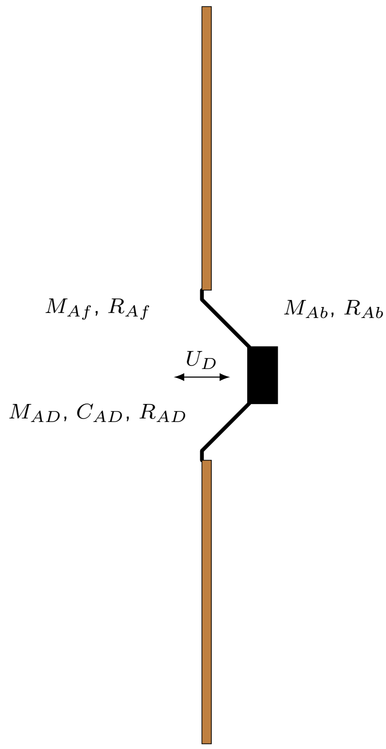


Figure 8.1: Infinite baffle enclosure.

Now that we have introduced appropriate sound radiation models we are in a position to begin modelling loudspeaker systems. We will begin by considering the simple case of an infinite baffle, before introducing sealed and vented cabinet designs.

8.1 Infinite Baffle

The infinite baffle is the simplest loudspeaker system to consider. Although the construction of an infinite baffle is somewhat impractical, a decent approximation can be obtained by housing a loudspeaker driver in a large wooden panel, placed in an anechoic room. In fact, it is this sort of configuration that loudspeaker drivers are typically characterised in.

When placed in an infinite baffle the loudspeaker driver is loaded by the acoustic free space at the front and rear. We have derived this acoustic loading for a radiating piston in the low frequency limit, see equation 7.20. The two terms in equation 7.20 can be associated with the resistive and inertial effects of the air surrounding the loudspeaker. In our equivalent circuit these effects can be modelled using an equivalent resistor and inductor, as in figure 8.2.

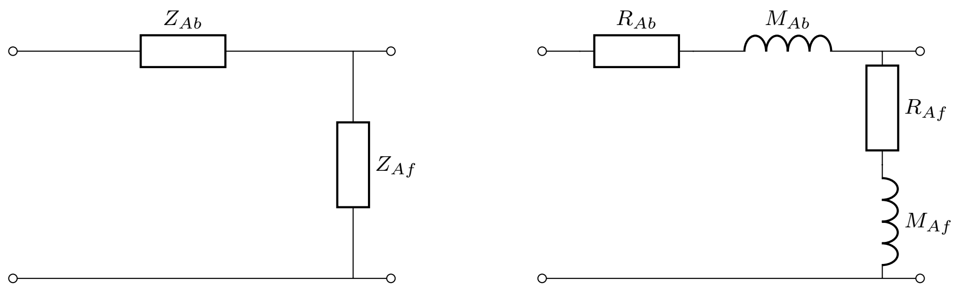


Figure 8.2: For an infinite baffle the arbitrary acoustic loads are replaced by an equivalent resistor and inductor.

Noting that the radiation load on the front (Af) and rear (Ab) are identical, we have

$$R_{Ab} = R_{Af} = \frac{\rho_0 c k^2}{2\pi} \tag{8.1}$$

and

$$M_{Ab} = M_{Af} = \frac{8\rho_0}{3\pi^2 a}. \tag{8.2}$$

Substituting the infinite baffle radiation load (see figure 8.2) into the full equivalent circuit model (see figure 6.12) leads to the circuit in figure 8.3. This circuit models the low frequency behaviour of a loudspeaker in an infinite baffle.

Recall that at low frequencies the impedance of a capacitor is high. Consequently, across the parallel RC section the current will flow primarily through the

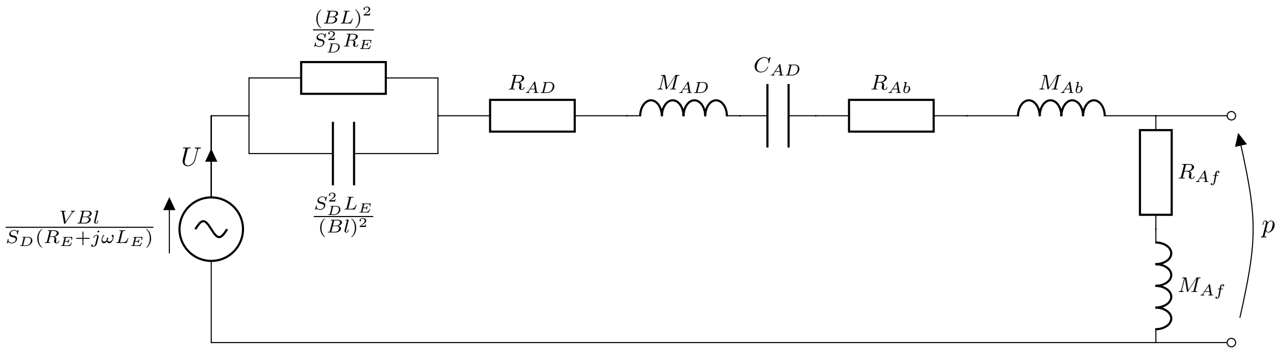


Figure 8.3: Complete equivalent circuit including radiation loading due to infinite baffle.

resistor element, allowing us to neglect the parallel capacitance. Similarly, at low frequencies the contribution of $j\omega L_E$ to the voltage source will be negligible compared to that of the DC resistance R_E. With the above in mind figure 8.3 can be simplified to that of figure 8.4 (which now neglects the voice coil inductance L_E).

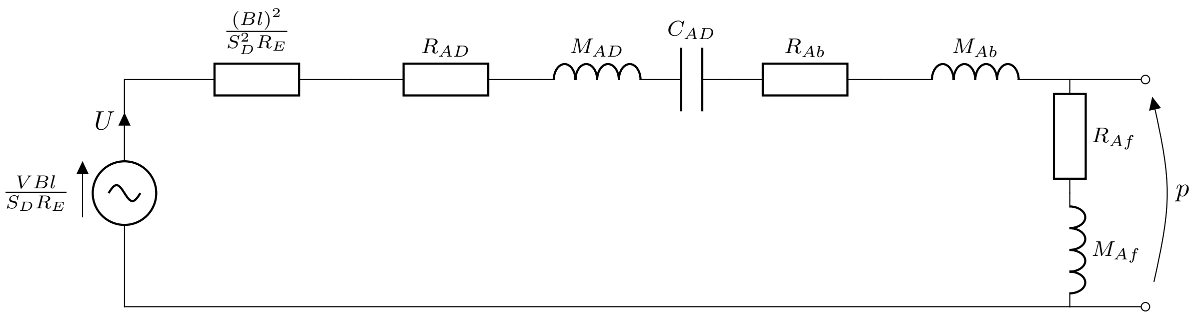


Figure 8.4: Simplified complete equivalent circuit including radiation loading due to infinite baffle.

For ease of analysis it is convenient to further simplify figure 8.3 by gathering like terms. For the infinite baffle loudspeaker we define the *total 'speaker' damping* by adding together the damping contributions from the electrical ($(Bl)^2/S_D^2 R_E$), mechanical (R_{AD}), and acoustic domains ($2R_{Af}$),

$$R_{AS} = \frac{(Bl)^2}{S_D^2 R_E} + R_{AD} + 2R_{Af}. \tag{8.3}$$

As above, we can define the *total speaker mass* as,

$$M_{AS} = M_{AD} + 2M_{Af}. \tag{8.4}$$

Finally, we have the *total speaker compliance*,

$$C_{AS} = C_{AD}. \tag{8.5}$$

More generally we can define the *total acoustic* mass, damping and compliance, M_{AT}, R_{AT} and C_{AT}, which describe the total mass, damping and compliance in a *loudspeaker system*. For an infinite baffle loudspeaker the only contributions to the mass, damping and compliance are from the speaker itself, and so $M_{AT} = M_{AS}$, $R_{AT} = R_{AS}$ and $C_{AT} = C_{AS}$. This will not be the case for a sealed or vented design.

Note that by adding an infinite baffle and acoustic loading we introduce addition mass and damping. The compliance however is unaffected; the total acoustic

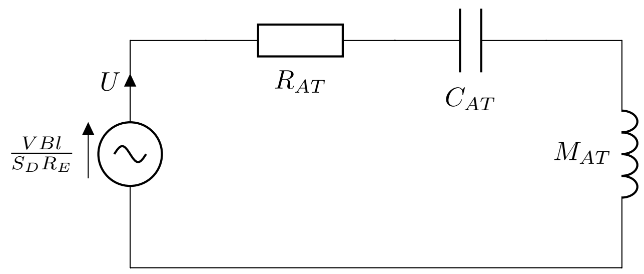


Figure 8.5: Complete equivalent circuit including radiation load using *total acoustic quantities*.

compliance is simply the mechanical suspensions compliance in the acoustic domain. Using the newly defined *total acoustic* mass, compliance and damping, a new equivalent circuit can be formed as in figure 8.5.

To predict a baffled loudspeaker's radiated sound we need to first determine the driver's volume velocity. This is then substituted into the rigid piston radiation model (equation 7.14) to predict the radiated pressure. To determine the driver's volume velocity we simply note that our equivalent circuit has been formulated according to the impedance analogy. As such, volume velocity is analogous to current. Find the volume velocity is therefore equivalent to finding the current through our equivalent circuit.

From Ohm's Law we know that $V = iZ$, or,

$$\frac{VBl}{S_D R_E} = U Z_T \tag{8.6}$$

where Z_T is the total impedance of the circuit. Since figure 8.5 is a series RLC circuit the total impedance is a straightforward sum. The volume velocity is then given by,

$$U = \frac{\frac{VBl}{S_D R_E}}{R_{AT} + j\omega M_{AT} + \frac{1}{j\omega C_{AT}}}. \tag{8.7}$$

It is convenient to rearrange equation 8.7 by factoring out $j\omega M_{AT}$ from the denominator and grouping with the voltage term,

$$U = \frac{VBl}{S_D R_E j\omega M_{AT}} \left[\frac{1}{1 + \frac{R_{AT}}{j\omega M_{AT}} + \frac{1}{j\omega j\omega M_{AT} C_{AT}}} \right]. \tag{8.8}$$

Equation 8.8 can be parametrised by recalling the relations, $\omega_s^2 = 1/M_{AS}C_{AS}$ and $Q_{TS}/\omega_c = M_{AS}/R_{AS}$ from our analysis of RLC circuits and mass spring systems,

$$U = \frac{VBl}{S_D R_E j\omega M_{AS}} \left[\frac{1}{1 + \frac{1}{Q_{TS}}\left(\frac{\omega_s}{j\omega}\right) + \left(\frac{\omega_s}{j\omega}\right)^2} \right]. \tag{8.9}$$

The volume velocity is now expressed as the product of two terms. The first term is inversely proportional to frequency and looks similar to first order low-pass filter response. The second term corresponds to a 2nd order high pass filter. For convenience we can define,

$$E(j\omega) = \left[\frac{1}{1 + \frac{1}{Q_{TS}}\left(\frac{\omega_s}{j\omega}\right) + \left(\frac{\omega_s}{j\omega}\right)^2} \right]. \tag{8.10}$$

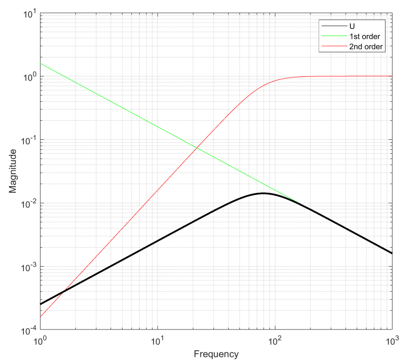


Figure 8.6: Volume velocity (black) expressed as the product of two terms: first order low pass filter (green) and second order high pass filter (red).

The characteristics of the low pass (green) and high pass (red) terms in equation 8.9 are shown in figure 8.6. When multiplied together they yield a volume velocity with a resonant response shape, as expected given that the driver is modelled as a mass on a spring. The volume velocity in figure 8.6 has been modelled using a Butterworth response with $Q_{TS} = 0.707$.

The shape of the filter response $E(j\omega)$ depends entirely on the Q-factor Q_{TS} and the resonance frequency ω_s. The Q-factor Q_{TS} corresponds to that of the baffled loudspeaker, and from figure 8.5 is given by,

$$Q_{TS} = \left(\frac{(Bl)^2}{S_D^2 R_E} + R_{ASm}\right)^{-1} \sqrt{\frac{M_{AS}}{C_{AS}}} \tag{8.11}$$

or

$$Q_{TS} = \omega_s \frac{M_{AS}}{R_{AS}} \tag{8.12}$$

where $R_{ASm} = R_{AD} + 2R_{Af}$ is used to denote the *acoustic* damping due to the *speaker* (i.e. the mechanical damping and air load, neglecting the electrical damping), and R_{AS} is the speakers total acoustic damping including the electrical damping. It is often convenient to express the total Q-factor in terms of the electrical and mechanical/acoustic damping separately. This can be done by defining the *mechanical* and *electrical* Q-factors,

$$Q_{TM} = \frac{1}{R_{ASm}} \sqrt{\frac{M_{AS}}{C_{AS}}} \tag{8.13}$$

and

$$Q_{ES} = \left(\frac{(Bl)^2}{S_D^2 R_E}\right)^{-1} \sqrt{\frac{M_{AS}}{C_{AS}}}. \tag{8.14}$$

Note that Q_{TS} includes the effect of air loading on the driver. The total Q-factor, as in equation 8.11, can then calculated using the produce over sum rule as so,

$$Q_{TS} = \frac{Q_{TM} Q_{ES}}{Q_{TM} + Q_{ES}}. \tag{8.15}$$

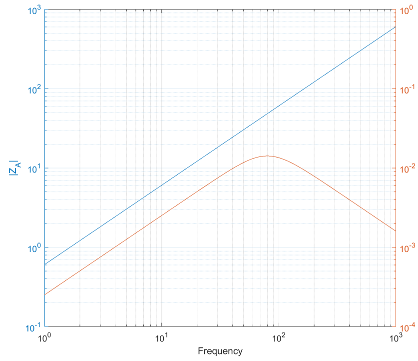


Figure 8.7: Volume velocity (orange) vs. radiation impedance of piston (blue)

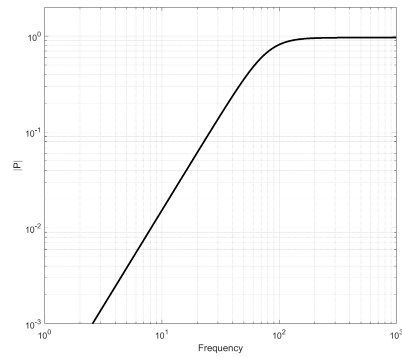


Figure 8.8: Radiated pressure from infinite baffle.

8.1.1 Velocity to Pressure Conversion

Having determined the driver's volume velocity (see equation 8.9) we are now able to predict the radiated acoustic pressure using the piston equation,

$$p(r, \theta, t) = \frac{j\rho_0 \omega U}{2\pi r} e^{j(\omega t - kr)} \left[\frac{2J_1(ka \sin\theta)}{ka \sin\theta}\right] \tag{8.16}$$

where we have substituted the volume velocity $U = \pi a^2 u$ in place of the surface velocity u, and replaced $k = \omega/c$. Assuming an on-axis response (so we can ignore the directivity factor) and substituting in the volume velocity we obtain,

$$p(r, \theta, t) = \frac{j\rho_0 \omega}{2\pi r} \frac{VBl}{S_D R_E j\omega M_{AS}} E(j\omega). \tag{8.17}$$

The radiation impedance ($j\rho_0\omega/2\pi r$) and volume velocity terms in equation 8.17 are shown in figure 8.7. Notice that below the resonant frequency of the driver both terms increase with a 6dB/oct slope. Above resonance the volume velocity begins to decrease with a -6dB/oct slope. When combined this downward sloping volume velocity counter acts the upwards sloping radiation impedance, leaving a flat frequency response. This is exactly what we want from a loudspeaker. Below the driver resonance the two slopes combine to give a 12 dB/oct slope.

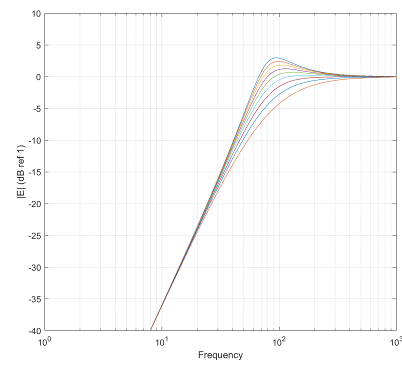


Figure 8.9: Frequency response in dB of an infinite baffle loudspeaker with varying Q-factors.

Notice that the frequency variable ω in the piston radiation term and initial volume velocity term will cancel with one another,

$$p(r,\theta,t) = \frac{\rho_0 V Bl}{2\pi r S_D R_E M_{AS}} E(j\omega). \tag{8.18}$$

Consequently, the only frequency dependent term in equation 8.18 is that of the 2nd order high pass filter $E(j\omega)$. The remaining terms are simply constants associated with the sensitivity of the loudspeaker (e.g. if we increase the mass M_{AS}, we decrease the radiated pressure). Hence, to evaluate the frequency response of a loudspeaker we only need to consider the high pass filter $E(j\omega)$.

Noting that the flat response region occurs only after the driver's resonant frequency (as the volume velocity begins to decrease with increasing frequency) we arrive at our first important design concept - *the lower the driver's resonant frequency the lower the flat response region extends.*

It is important to remember that figure 8.8 represents just one of many available response shapes. Shown in figure 8.9 are the response shapes obtained for a Q_{TS} varying between 0.5 and 1.3 in increments of 0.1. The effect of Q-factor on loudspeaker performance will be treated in more detail when we consider sealed and vented cabinets.

8.1.2 Electrical Impedance

Before we move onto the sealed cabinet loudspeaker lets consider the electrical impedance of a loudspeaker in an infinite baffle. Recall our equivalent circuit in figure 6.5. To determine the acoustic impedance (and from it the volume velocity) of our loudspeaker we referred all components across to the acoustic domain. To do this we scaled the electrical and mechanical elements by $1/(Bl)^2$ and S_D^2. To obtain the electrical impedance we must refer all components across to the electrical domain. This means scaling the acoustic and mechanical domain by $1/S_D^2$ and $(Bl)^2$. The resulting circuit is shown in figure 8.10.

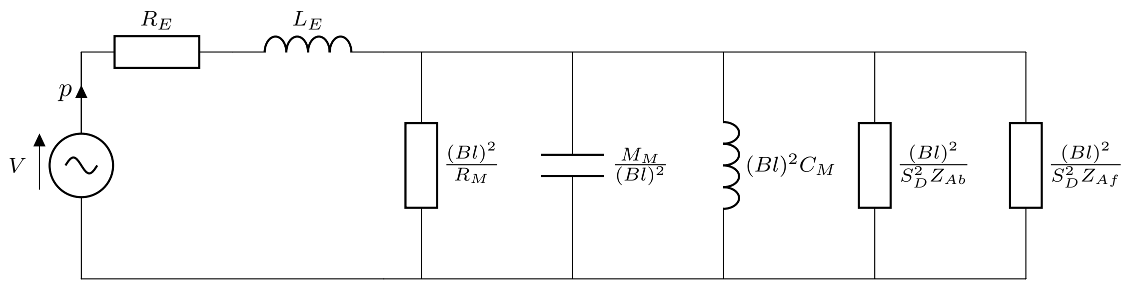


Figure 8.10: Equivalent circuit with all elements referred to the electrical domain.

To determine the electrical impedance of the loudspeaker driver we must find the total impedance of figure 8.10. To do so it is convenient to group together the electrical and mechanical/acoustic elements such that the total electrical impedance is given by,

$$Z_{ET} = Z_E + Z_{MS} \tag{8.19}$$

where,

$$Z_E = R_E + j\omega L_E \tag{8.20}$$

and

$$Z_{MS} = \left(\frac{1}{\frac{(Bl)^2}{j\omega M_M}} + \frac{1}{\frac{(Bl)^2}{R_M}} + \frac{1}{j\omega C_M (Bl)^2} + \frac{1}{\frac{1}{2}\frac{(Bl)^2}{S_D^2 j\omega M_{Af}}} + \frac{1}{\frac{1}{2}\frac{(Bl)^2}{S_D^2 j\omega R_{Af}}} \right)^{-1}. \tag{8.21}$$

Note that we are considering an infinite baffle loading, and so the front and rear acoustic loads are the same. Since they are in parallel they can be combined to a single elements with half the impedance. The acoustic mass and damping can then be combined with the mechanical elements as so: $M_{MT} = M_M + 2S_D^2 M_{Af}$, and $R_{MT} = R_M + 2S_D^2 R_{Af}$. Equation 8.21 can now be rewritten as,

$$Z_{MS} = \left(\frac{j\omega M_{MT}}{(Bl)^2} + \frac{R_{MT}}{(Bl)^2} + \frac{1}{j\omega C_M (Bl)^2} \right)^{-1}. \tag{8.22}$$

We are interested in rearranging the above into a more convenient form. We start by factoring $R_{MT}/(Bl)^2$ out of the denominator,

$$Z_{MS} = \frac{1}{\left(\frac{j\omega M_{MT}}{R_{MT}} + 1 + \frac{1}{j\omega C_M R_{MT}} \right) \frac{R_{MT}}{(Bl)^2}}. \tag{8.23}$$

Defining the new variable $R_{ES} = (Bl)^2/R_{MT}$, and multiplying top and bottom by $j\omega C_M R_{MT}$, we obtain,

$$Z_{MS} = R_{ES} \frac{1}{\left(\frac{j\omega M_{MT}}{R_{MT}} + 1 + \frac{1}{j\omega C_M R_{MT}} \right)} \frac{j\omega C_M R_{MT}}{j\omega C_M R_{MT}}, \tag{8.24}$$

and subsequently, Defining the new variable $R_{ES} = (Bl)^2/R_{MT}$, and multiplying top and bottom by $j\omega C_M R_{MT}$, we obtain,

$$Z_{MS} = R_{ES} \frac{j\omega C_M R_{MT}}{((j\omega)^2 M_{MT} C_M + j\omega C_M R_{MT} + 1)}. \tag{8.25}$$

Now we recall that:

$$\omega_c^2 = 1/C_M M_{MT} \tag{8.26}$$

and

$$Q_{MS}/\omega_c = M_{MT}/R_{MT} \rightarrow Q_{MS}\omega_c = 1/C_M R_{MT}. \tag{8.27}$$

Substitution into Z_{MS} then yields,

$$Z_{MS} = R_{ES} \frac{\frac{j\omega}{Q_{MS}\omega_c}}{\left(\frac{(j\omega)^2}{\omega_c^2} + \frac{j\omega}{Q_{MS}\omega_c} + 1 \right)}. \tag{8.28}$$

From equations 8.19, 8.20 and 8.28 total electrical impedance of the a loudspeaker driver (in an infinite baffle) is given by,

$$Z_{ET} = R_E + j\omega L_E + R_{ES} \frac{\frac{j\omega}{Q_{MS}\omega_c}}{\left(\frac{(j\omega)^2}{\omega_c^2} + \frac{j\omega}{Q_{MS}\omega_c} + 1 \right)}. \tag{8.29}$$

The electrical impedance from equation 8.29 is plotted in figure 8.11. It exhibits all of the features we see in an experimentally obtained impedance. At high frequencies the voice coil inductance L_E becomes the dominant source of impedance (note that it was the inductor impedance that we neglected between figures 8.3 and 8.4), whilst at very low frequencies the only term remaining is the voice coil

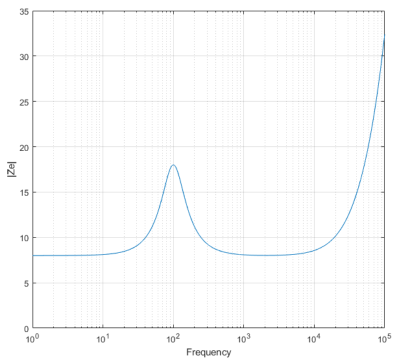


Figure 8.11: Electrical impedance of a moving coil loudspeaker in an infinite baffle.

resistance R_E (i.e. the nominal DC resistance of the loudspeaker). In the vicinity of the driver resonance a peak in the electrical impedance is observed due to the rightmost term. Exactly at resonance ($\omega = \omega_c$) this complex term reduces to 1, and the total impedance is $Z_{ET} \approx R_E + R_{ES}$.

Note that the peak electrical impedance occurs at the driver resonance, i.e. when the mechanical impedance is a minimum. At resonance the driver velocity is a maximum (because its impedance is small). The large driver velocity induces a back EMF in the voice coil that counters the applied current, and so the electrical impedance is increased.

8.1.3 Theile-Small Parameters

In order to use our equivalent circuit and model the behaviour of a loudspeaker we need to know its electrical, mechanical and acoustical properties. In particular, from figure 8.10 we can identify the following parameters:

- M_{MT} Mass of the diaphragm/coil (including acoustic load) in kilograms
- R_{MT} The mechanical resistance of a driver's suspension (including acoustic load)
- C_M Compliance of the driver's suspension, in meters per newton
- S_D Projected area of the driver diaphragm, in square meters
- L_E Voice coil inductance measured in millihenries
- R_E DC resistance of the voice coil, measured in ohms
- Bl The product of magnet field strength in the voice coil gap and the length of wire in the magnetic field

These fundamental parameters are often quite hard to measure directly. Many of the above quantities are related through other quantities, such as resonant frequency ω_c and Q-factor Q_{TS}, which are directly measurable. Based on this it is convenient to characterise the low-frequency properties of a loudspeaker by what are known as the Theile-Small parameters (named after Neville Theile and Richard Small; two pioneers of loudspeaker design). The Theile-Small parameters are:

- R_E Nominal DC resistance
- Q_{ES} Electrical Q-factor (due to electrical damping only)
- Q_{TM} Mechanical Q-factor (due to the mechanical damping/acoustic loading only)
- f_s Free suspension driver resonance
- S_D Driver's effective surface area
- V_{AS} Equivalent suspension volume

where V_{AS} represents the volume of air having the same compliance as the suspension,

$$V_{AS} = C_{AD}\rho_0 c^2 = C_M S_D^2 \rho_0 c^2. \tag{8.30}$$

Based on the above Theile-Small parameters we can recover any of the fundamental loudspeaker parameters using the equations derived above. For example, the mechanical compliance can be obtained by,

$$C_M = \frac{V_{AS}}{S_D^2 \rho_0 c^2}. \tag{8.31}$$

Once the mechanical compliance is known, M_{MT} can be found by,

$$M_{MT} = \frac{1}{\omega_s^2 C_M}. \tag{8.32}$$

The mechanical damping R_{MT} can then be found from the mass, compliance and mechanical Q-factor,

$$R_{MT} = \frac{1}{Q_{MS}}\sqrt{\frac{M_{MT}}{C_M}}. \tag{8.33}$$

Similarly the force factor Bl can be found as,

$$(Bl)^2 = \frac{R_E}{Q_{ES}}\sqrt{\frac{M_{MT}}{C_M}}. \tag{8.34}$$

The measurement of the Theile-Small parameters is covered as part of your first Loudspeaker Lab. For more details please see the lab handbook.

8.2 Sealed Cabinet

So far we have only considered loudspeakers mounted in an infinite baffle. Clearly, infinite baffles are somewhat impractical (expensive and take up a lot of room). Recall that the purpose of the infinite baffle was simply to separate the front and rear radiated sound. An alternative approach is to simply mount the loudspeaker in a sealed cabinet. This way the rear radiated sound is kept within the cabinet, and any destructive interference avoided.

The sealed cabinet is the simplest loudspeaker design. They were first introduced in the 1940's and by the 1950's they had become a popular choice in HiFi. A key feature of a sealed cabinet design is that they use a relatively compliant loudspeaker driver. This enables the sealed cabinet itself to control the loudspeaker response; the cabinet adds a stiffness to the mass-spring behaviour of the loudspeaker driver. To design a sealed cabinet loudspeaker we must modify our infinite baffle loudspeaker circuit (see figure 8.3) to account for the modified rear acoustic load (due to the cabinet).

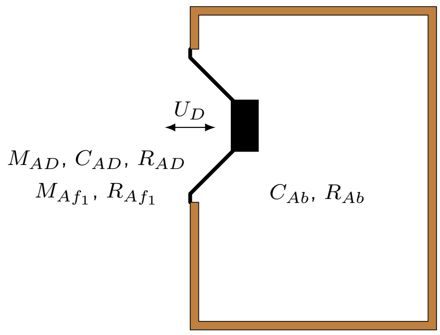


Figure 8.12: Sealed enclosure.

We saw previously that a cavity, or sealed volume of air, possess an acoustic compliance. By introducing a sealed cabinet, the loudspeaker's rear acoustic load must now contain a compliant term that describes the 'springiness' of the air cavity, alongside any additional inertial effects. As in the infinite baffle case there will also be an inertial and resistive contribution from the air loading within the cabinet. A further additional damping will be introduced if any damping material (e.g. mineral wool, foam, etc.) is put in the cavity. As such, the rear acoustic load Z_{Ab} is given by,

$$Z_{Ab} = R_{Ab} + j\omega M_{Ab} + R_{AB} + j\omega M_{AB} + \frac{1}{j\omega C_{AB}} \tag{8.35}$$

where R_{Ab} and M_{Ab} describe the resistive and inertial air loading, and R_{AB}, M_{AB} and C_{AB} describe the effect of the cabinet.

The added resistance due to damping material in the cavity R_{AB} is often difficult to quantify and so, for now, we can ignore it. A further simplification can be made by noting that for a small volume, such as a sealed cabinet, the impedance contribution of the compliance C_{AB} is far greater than that of any additional inertial loading M_{AB},

$$\frac{1}{j\omega C_{AB}} = \frac{\rho_0 c^2}{j\omega V_B} >> j\omega M_{AB}. \tag{8.36}$$

We can therefore neglect the inertial and resistive cabinet load, and approximate the rear loading as a capacitance in series with the infinite baffle load, as in

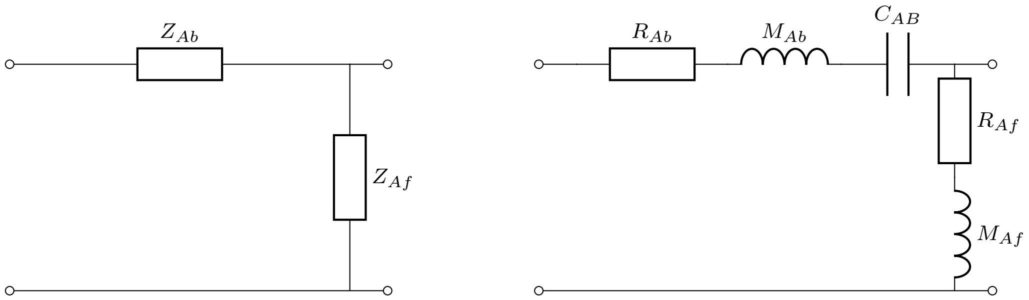


Figure 8.13: Front and rear acoustic loading for a sealed cabinet

figure 8.13. Note that a capitalised B subscript is used here to denote the *box* compliance. Based on the above approximation, our full equivalent circuit becomes that of figure 8.14.

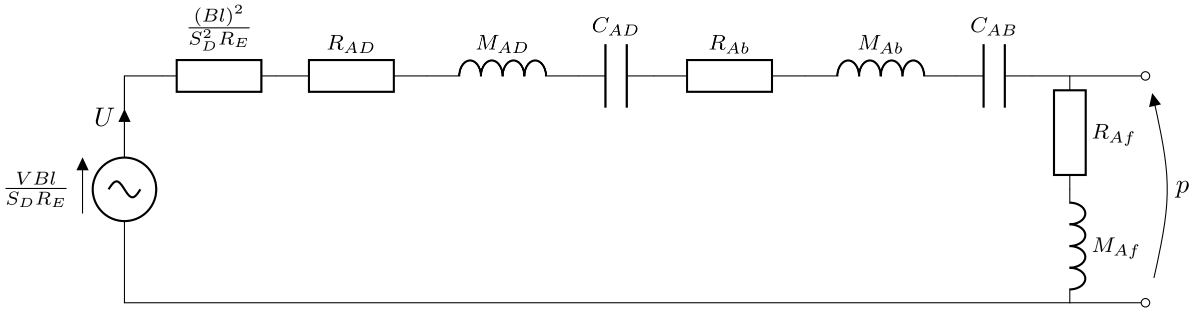


Figure 8.14: Complete equivalent circuit of a sealed cabinet loudspeaker including front and rear radiation loading.

By grouping like terms figure 8.14 may be reduced to the simple RLC circuit shown in figure 8.15. Note that this circuit is identical in form to that of figure 8.5 (i.e. the infinite baffle case), the only difference are the values of the total acoustic components.

The introduction of a sealed cabinet, assuming no additional damping is introduced, has a negligible effect on the resistive and inertial loading of the loudspeaker. The greatest change comes from the additional compliance of the cavity. As such, we can assume that the total acoustic mass and damping is the same as in the infinite baffle case, i.e.,

$$M_{AT} \approx M_{AS} \tag{8.37}$$

and

$$R_{AT} \approx R_{AS}. \tag{8.38}$$

The total acoustic compliance however, has changed considerably. From figure 8.14 we have two series compliances. Using the product over sum rule their combined compliance is,

$$C_{AT} = \frac{C_{AS}C_{AB}}{C_{AS} + C_{AB}} \tag{8.39}$$

where $C_{AS} = C_{AD}$ is the compliance of the mechanical suspension (in acoustic units), and C_{AB} is the acoustic compliance of the sealed cabinet, calculated as per equation 5.25.

Although we have decided to ignore the effect of added cabinet damping, it is important to acknowledge its effect on the response of a sealed cabinet loudspeaker. In general, an increase in the damping of a system decreases the Q-factor of its resonance. This will lead to a reduced level at the low frequency cut-off and a more gradual roll-off in its frequency response.

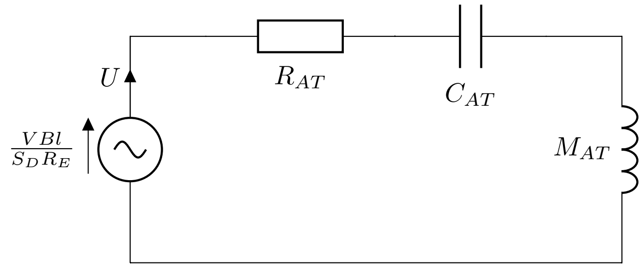


Figure 8.15: Complete equivalent circuit including radiation load using *total acoustic quantities*.

From figure 8.15 we can identify the volume velocity (equivalent to current) as,

$$U = \frac{\frac{VBl}{S_D R_E}}{R_{AT} + j\omega M_{AT} + \frac{1}{j\omega C_{AT}}}. \tag{8.40}$$

Given that figure 8.15 is identical in form to figure 8.5, it is of no surprise that the volume velocity of the sealed loudspeaker driver is of the same form as the infinite baffles. However, from equations 8.37-8.39 we can see that, unlike the total acoustic mass and damping, the total acoustic compliance is different to that of an infinite baffle. The added cabinet compliance will change the resonant frequency of the system,

$$\omega_c = \sqrt{\frac{1}{M_{AT} C_{AT}}} \approx \sqrt{\frac{1}{M_{AS} C_{AT}}} \tag{8.41}$$

and consequently the Q-factor since,

$$Q_{TC} = \frac{1}{R_{AT}} \sqrt{\frac{M_{AT}}{C_{AT}}} \approx \frac{1}{R_{AS}} \sqrt{\frac{M_{AS}}{C_{AT}}} \tag{8.42}$$

or

$$Q_{TC} = \frac{\omega_c^2 M_{AT}}{R_{AT}} \approx \frac{\omega_c^2 M_{AS}}{R_{AS}}. \tag{8.43}$$

Note that ω_c and Q_{TC} are the resonant frequency and Q-factor of the *sealed cabinet*, and that ω_s and Q_{TS} are the resonant frequency and Q-factor of the driver in an infinite baffle (i.e. in free space).

Equation 8.43 can be used to derive a very useful relation for sealed loudspeakers. If we divide both sides by ω_c^2,

$$\frac{Q_{TC}}{\omega_c} \approx \frac{M_{AS}}{R_{AS}}. \tag{8.44}$$

and recall the Q-factor of an infinite baffle loudspeaker (see equation 8.12),

$$\frac{Q_{TS}}{\omega_s} = \frac{M_{AS}}{R_{AS}}, \tag{8.45}$$

it is clear that,

$$\frac{Q_{TC}}{\omega_c} \approx \frac{Q_{TS}}{\omega_s}. \tag{8.46}$$

Equation 8.46 relates the resonant frequency and Q-factor of a free driver, and a sealed cabinet. Using this equation we can obtain the Q-factor of a sealed cabinet loudspeaker, provide the driver parameters and the resonance frequency of the sealed cabinet (from equation 8.41) are known.

Like the infinite baffle volume velocity, equation 8.40 can be parametrised as,

$$U = \frac{VBl}{S_D R_E j\omega M_{AT}} E(j\omega) \tag{8.47}$$

where the frequency response term $E(j\omega)$ is given by,

$$E(j\omega) = \left[\frac{1}{1 + \frac{1}{Q_{TC}}\left(\frac{\omega_c}{j\omega}\right) + \left(\frac{\omega_c}{j\omega}\right)^2} \right]. \tag{8.48}$$

Equation 8.47 can then be substituted into equation 7.14 for piston radiation. In doing so we observe the same first order high pass/low pass cancellation, leading to the radiated pressure,

$$p(r, \theta, t) = \frac{\rho_0 V B l}{2\pi r S_D R_E M_{AT}} E(j\omega). \tag{8.49}$$

Like equation 8.18, the only frequency dependent term in equation 8.49 is the frequency response $E(j\omega)$. The remaining terms are constants related to the sensitivity of the loudspeaker.

The process of designing a sealed cabinet loudspeaker involves choosing a particular response shape for $E(j\omega)$ (characterised by its two parameters ω_c and Q_{TC}), and then finding an appropriate driver and/or cabinet to achieve this response shape. In some cases the driver will be specified before hand and then the appropriate cabinet must be found. Alternatively, the cabinet design may be restricted to a particular volume, and so an appropriate driver must be found. The process of designing a sealed cabinet loudspeaker is often referred to as 'choosing an alignment'.

8.2.1 Choosing an Alignment

Shown in figure 8.16 are a selection of the possible frequency response shapes available from a sealed cabinet design. The response shapes shown are for a fixed $f_c \approx 80$ Hz with a Q_{TC} varying between 0.4 and 2. There are two important regions in the response shape: the cut-off and the reference region. The reference region is the region where the response shape flattens out and tends to a constant value. This is the region we want out loudspeaker to operate in as it will give us the flattest overall frequency response. The cut-off region lies between the reference region and the low frequency roll off. When choosing an alignment it

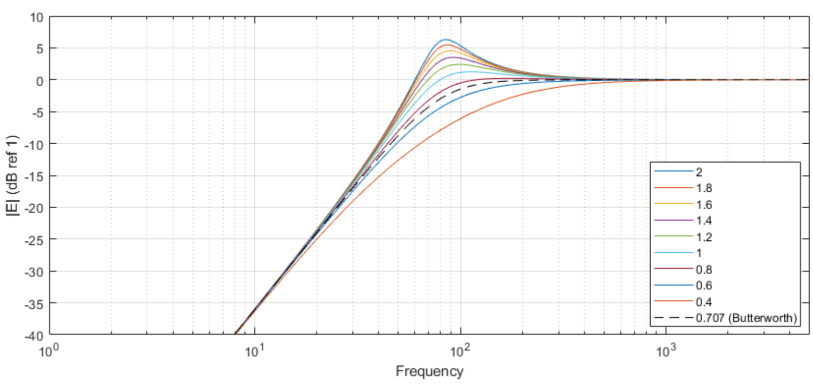


Figure 8.16: Frequency response shape for a sealed cabinet with $f_c \approx 80Hz$ and Q_{TC} varying between 0.4 and 2.

is the cut-off region that we control over. We are able to move it up and down the frequency axis (e.g. by making the box stiffer/more compliant or the driver heavier/lighter), but also increase or decrease its amplitude (by adjusting the amount of damping and therefore the Q-factor). The transition from reference region to -12 dB/oct roll-off depends on the characteristics of the cut-off region. A quicker transition is obtained when the Q-factor is higher, albeit at the cost of a low frequency boost. This boost, although undesirable if we are trying to achieve a flat frequency response, can be used to increase the perceived bass response of smaller loudspeakers. A popular, and worth remembering, Q-factor is that of the Butterworth alignment, $Q_{TC} = 0.707$.

The Butterworth alignment is also called the *maximally flat* frequency response. It is the flattest possible frequency response we can achieve for our sealed cabinet loudspeaker. If the Q-factor is increased by any amount ($Q_{TC} > 0.707$), the frequency response will have a gain greater than 0. Decreasing the Q-factor by any amount ($Q_{TC} > 0.707$) will introduce a more gradual roll-off.

It is important to note that whilst our equivalent circuit model will predict the reference region extending to infinity, this is not what happens in reality. Remember, our entire equivalent circuit approach is based on the lumped parameter assumption. At high frequencies this is no longer applicable (we start to get wave motion, e.g. cone break-up). Furthermore, we have chosen to neglected the influence of the voice coil's inductance, and are modelling radiation impedance using a first order approximation. As such, we are only able to rely on our equivalent circuit up to the limit of $ka \approx 1$.

8.2.1.1 Transient Response

It is important when choosing an alignment to remember that the Q-factor not only effects the frequency response shape, but also the system's transient response. A high Q-factor system will 'ring on' past the any transient attack, a low Q-factor system will decay much faster. Shown in figure 8.17 are the (normalised) transient responses of a loudspeaker driver with varying Q-factors. Note that once

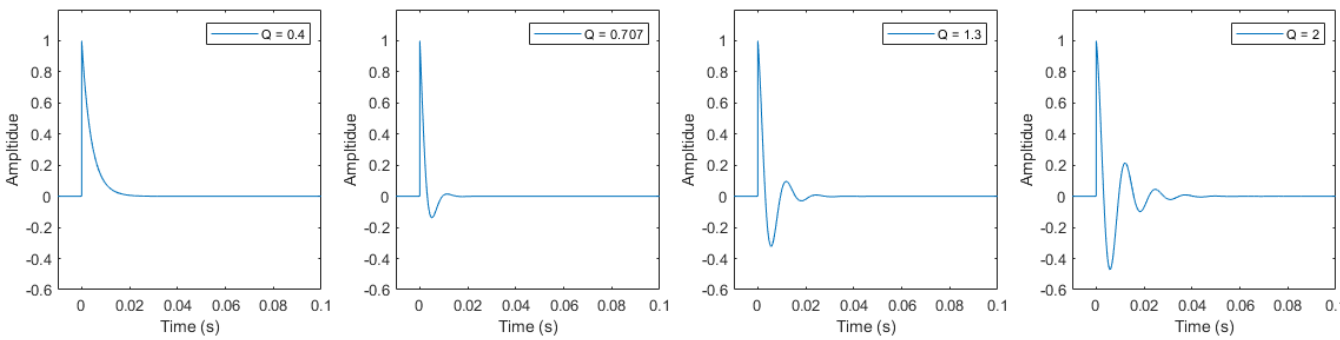


Figure 8.17: Transient response of a sealed cabinet loudspeaker for different Q-factors

the Q-factor goes below 0.5, the system becomes over damped and no oscillation is observed. Achieving a Q-factor of 0.5 (critically damped) will give the fastest possible transient response, but at the cost of a poorer frequency response. If the Q-factor obtained is quite large, say 1.4, we may get a superior frequency response, but we also get a longer transient decay. This prolonged decay is mostly producing one single note (corresponding to the peak in our frequency response). We don't really want this. A common Q-factor for a sealed cabinet is around 1. This gives us a best of both worlds.

8.2.1.2 Infinite Baffle vs. Sealed Cabinet

Here is a quick recap/comparison of the infinite baffle loudspeaker vs. its sealed cabinet counter part.

- Infinite Baffle:
  - $M_{AT} = M_{AS} = M_{AD} + 2M_{Af}$ - The total acoustic mass is the sum of the driver mass (in acoustic units) and the front and read inertial loading (both the same).
  - $C_{AT} = C_{AS} = C_{AD}$ - The total acoustic compliance is entirely that of the mechanical suspension (in acoustic units).
  - $\omega_s = \sqrt{\frac{1}{C_{AD}(M_{AD}+2M_{Af})}}$ - Free driver resonance.
  - $Q_{TS} = \frac{\omega_s M_{AS}}{R_{AS}}$ - Free driver Q-factor.
- Sealed Cabinet:
  - $M_{AT} = M_{AD} + M_{Af} + M_{Ab}$ - The total acoustic mass is the sum of the driver mass (in acoustic units) and the front and read inertial loading (not necessarily the same, often we ignore the rear inertial loading).
  - $C_{AT} = \frac{C_{AD}C_{AB}}{C_{AD}+C_{AB}}$ - The total acoustic compliance is the combined effect of the mechanical suspension (in acoustic units) and the cavity compliance.
  - $\omega_c = \sqrt{\frac{1}{\frac{C_{AD}C_{AB}}{C_{AD}+C_{AB}}(M_{AD}+M_{Af}+M_{Ab})}}$ - Sealed cabinet driver resonance.
  - $Q_{TC} = \frac{\omega_c M_{AT}}{R_{AT}}$ - Sealed cabinet Q-factor.

Under the assumptions that a) the change in inertial loading between the infinite baffle and sealed cabinet is negligible ($M_{AT} = M_{AS}$), and b) that the cabinet doesn't add any additional damping ($R_{AT} = R_{AS}$), the sealed and infinite baffle Q-factor and resonant frequencies as related through the equation,

$$\frac{Q_{TC}}{\omega_c} \approx \frac{Q_{TS}}{\omega_s}. \tag{8.50}$$

8.2.2 Example Design Procedure

The design procedure for a sealed cabinet will be demonstrated by way of an example. Suppose we are supplied with a driver whose Thiele-Small parameters are:

$$f_s = 45\ Hz \tag{8.51}$$

$$Q_{TS} = 0.35 \tag{8.52}$$

$$M_{MD} = 0.011\ kg \tag{8.53}$$

$$S_D = \pi \left(\frac{0.165}{2}\right)^2 = 0.021\ m^2 \tag{8.54}$$

What volume cabinet is required to achieve a Butterworth alignment? And what frequency is the sealed cabinet's resonant frequency?

From equation 8.46 we can calculate the coupled resonance as,

$$f_c = \frac{Q_{TC}}{Q_{TS}} f_s = \frac{0.707}{0.35} \times 45 = 90.9\ Hz. \tag{8.55}$$

The volume of a sealed cabinet V_{AS} is related to its compliance C_{AB} as per equation 8.30,

$$V = C_{AB}\rho_0 c^2. \tag{8.56}$$

To determine the box compliance we must first determine the total compliance required to obtain a resonant frequency of 90.9 Hz,

$$(2\pi 90.9)^2 = \frac{1}{M_{AT}C_{AT}} \rightarrow C_{AT} = \frac{1}{(2\pi 90.9)^2 \times 24.06} = 1.27 \cdot 10^{-7}\ m^5/N \tag{8.57}$$

where the total acoustic mass is calculated using (note that for simplicity we have chosen to ignore inertial air loading),

$$M_{AT} = \frac{M_{MD}}{S_D^2} = 24.06\ kg/m^4. \tag{8.58}$$

The equation for total acoustic compliance can now be rearranged to find the necessary box compliance,

$$C_{AT} = \frac{C_{AS}C_{AB}}{C_{AS}+C_{AB}} \rightarrow C_{AB} = \frac{C_{AS}C_{AT}}{C_{AS}-C_{AT}} = 1.68 \cdot 10^{-7}\ m^5/N, \tag{8.59}$$

where the driver's acoustic suspension compliance C_{AS} is obtained from the free driver resonance,

$$\omega_s^2 = \frac{1}{C_{AS}M_{AT}} \rightarrow C_{AS} = \frac{1}{\omega_s^2 M_{AT}} = 5.19 \cdot 10^{-7}\ m^5/N. \tag{8.60}$$

Substituting $C_{AT} = 1.68 \cdot 10^{-7}$ into equation 8.56 yields the volume,

$$(1.68 \cdot 10^{-7}) \times 1.21 \times 343^2 = 0.024\ m^3. \tag{8.61}$$

So, for the driver parameters specified, the cabinet volume required to achieve a Butterworth response is 24 L. The resonant frequency of this speaker will be 90.9 Hz.

The frequency range over which a loudspeaker is considered to have a flat frequency response is often specified from its lower -3dB point upwards. Once the Q-factor and resonant frequency of the sealed cabinet loudspeaker are known, the -3dB frequency can be determined as follows.

Note that the -3dB point corresponds to the frequency at which power is half its maximum value, $|E(j\omega)|^2 = 1/2$, or

$$\frac{1}{2} = \left| \frac{1}{1+\frac{1}{Q_{TC}}\left(\frac{\omega_c}{j\omega_{3dB}}\right)+\left(\frac{\omega_c}{j\omega_{3dB}}\right)^2} \right|^2. \tag{8.62}$$

Considering the equality between denominators,

$$2 = \left| 1+\frac{1}{Q_{TC}}\left(\frac{\omega_c}{j\omega_{3dB}}\right)+\left(\frac{\omega_c}{j\omega_{3dB}}\right)^2 \right|^2, \tag{8.63}$$

and evaluating the magnitude square of the above leads to,

$$2 = \left[1-\left(\frac{\omega_c}{\omega_{3dB}}\right)^2\right]^2 + \left[\frac{1}{Q_{TC}}\left(\frac{\omega_c}{\omega_{3dB}}\right)\right]^2. \tag{8.64}$$

On expanding the brackets we get,

$$1 = \left(\frac{1}{Q_{TC}^2} - 2\right)\left(\frac{\omega_c}{\omega_{3dB}}\right)^2 + \left(\frac{\omega_c}{\omega_{3dB}}\right)^4. \tag{8.65}$$

For a Butterworth alignment, $Q_{TC} = 0.707 = 1/\sqrt{2}$. Substituting this into the above leads to,

$$1 = \left(\frac{\omega_c}{\omega_{3dB}}\right)^4. \tag{8.66}$$

Taking the forth-root of both sides then yields,

$$\omega_{3dB} = \omega_c. \tag{8.67}$$

This is a unique property of the Butterworth alignment, the -3dB cut-off is equal to the resonant frequency. For other alignments the -3dB cut-off and resonant frequency do not coincide.

8.2.3 General design principles

It is important to note that some drivers just are'nt suitable for a sealed enclosure, for example because the box volume required is prohibitively large. There are some particular attributes that are needed for a sealed box. Here are some rules of thumb:

1) The compliance ratio ($\alpha = C_{AB}/C_{AS}$) should be more than 3, this ensures that the size of the box is not too large (a large volume V_{AB} gives us a small α)

2) The compliance ratio should be less than 10, this ensures that the mechanical suspension is not too compliant (a very compliant/flexible suspension gives us a large α). If the suspension is too compliant the system might become mechanically unstable. For a reasonable sized cabinet, a large α means a highly compliant driver.

3) The resonant frequency of the driver needs to be lower that the total resonance. When we add a cabinet we increase the stiffness and so increase the resonance frequency. In general for a sealed box design we want a driver with a low resonant frequency, so as to achieve a low system resonance. As a rule of thumb the free driver resonance should be less than half the desired system resonance (otherwise we will have to make the cabinet unrealistically small to achieve the necessary increase in stiffness)

Having a low resonant frequency means that we need a large cone mass. This reduces the total velocity amplitude (remember we factored out a mass term from our volume velocity). To compensate for this poor sensitivity we have to have long excursion limits so that the driver can displace a large enough volume of air at low frequencies.

Richard small suggested a good rule of thumb for determining if a driver could be used for sealed cabinet, he called it the efficiency bandwidth product. Its defined simply as the free driver resonance over the free driver Q factor,

$$EBP = \frac{f_s}{Q_{TS}}. \tag{8.68}$$

It's a single number that shows the trade-off between the efficiency and bandwidth of a driver. These two parameters are inversely related: for maximum possibly efficiency, the bandwidth must be narrow; for maximum possible bandwidth, the efficiency then becomes less.

An EBP of 50 or less indicates a low resonance compared with the Q factor. This is what we want for a sealed box driver. An EBP of around 100 is more suitable for vented cab (which will get an extended bass response due to the added vent!). A driver with an EBP between 50 and 100 can be used in either sealed or vented designs.

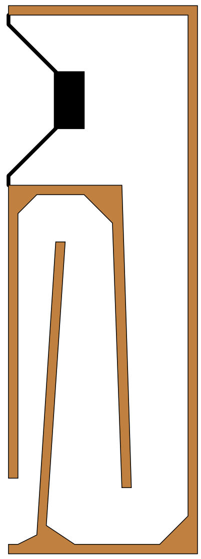


Figure 8.18: Transmission line enclosure.

8.3 Transmission Line

Before we move onto the vented cabinet design, we will very briefly consider what is called a transmission line enclosure. It turns out that we have already done all the hard work, and so the analysis of a transmission line loudspeaker is quite straightforward.

So what is the point behind a transmission line enclosure? In its simplest form, we replace the closed cavity of the sealed cabinet with a long duct (usually filled with some absorptive material). This duct is designed so that the rear radiation is completely dissipate so that a) there is no interference with the radiation from the main driver and b) there are no standing waves within the duct. How do we model this sort of enclosure?

The front loading is unchanged and so will be the same as our sealed cabinet. The rear loading however, is no longer that of a sealed cabinet. So we no longer have an acoustic compliance acting on the driver. Instead we have what appears to the driver as an infinite duct. We have already derived the impedance of an infinite duct, and see that it is purely resistive. So what effect does this have one our equivalent circuit model?

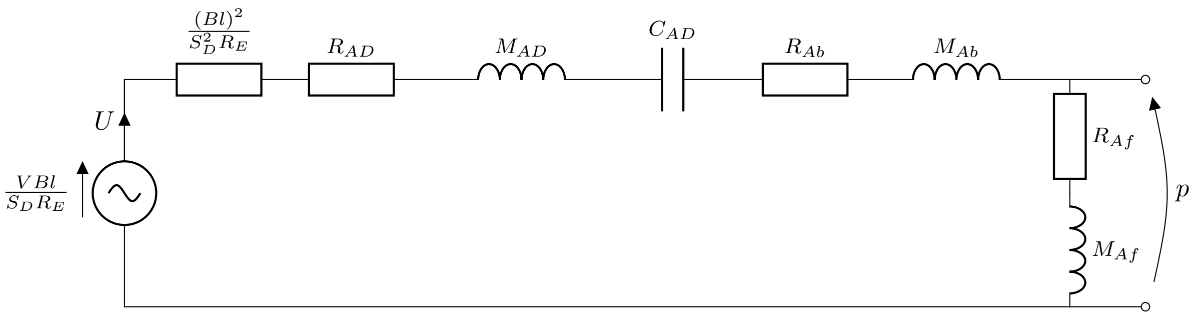


Figure 8.19: Complete equivalent circuit of a transmission line loudspeaker including front and rear radiation loading.

The capacitor representing the rear compliance has been removed, and the rear resistance R_{Ab} replaced by the resistance of an infinite duct,

$$R_{Ab} = \frac{\rho_0 c}{S}. \tag{8.69}$$

The resulting circuit is that of figure 8.19. Note that the impedance of an infinite duct is purely resistive, and so doesn't add any reactance (not additional mass or compliance), $M_{AT} = M_{AS}$ and $C_{AT} = C_{AS}$. Consequently, the resonant frequency of the transmission line loudspeaker is the same as the free driver's resonance,

$$\omega_c = \sqrt{\frac{1}{M_{AD} C_{AD}}} = \omega_s. \tag{8.70}$$

This means that we can get a better low frequency response than the sealed cabinet. However, it is essential that the absorbent material in the duct provides an anechoic like termination, otherwise we will get unwanted radiation from the rear of the driver. More complex designs can use this radiation to improve the response, but that is beyond the scope of this module. We will use a vented cabinet design to achieve this sort of improvement instead.

8.4 Vented Cabinet

So far we have covered the simplest loudspeaker enclosure, the sealed cabinet. We saw that its low frequency response was basically limited by the free driver resonance and the added compliance of the cabinet. Now lets ask the question on everyone's mind... 'what happens if we put a hole in the box?' This is what we called a vented box loudspeaker (also known as a bass-reflex loudspeaker).

People knew very early on that by putting a hole in the box we could improve the low frequency response of a loudspeaker. The problem was that there was no fool proof way of designing them. It was typically done by trial and error, using practical experience. So getting a good design was quite difficult.

So what are the benefits of having a vented enclosure? Well, what happens to the resonance of a loudspeaker when we put it in sealed box? It goes up. This is what limits the loudspeaker's low frequency performance. It turns out that this isn't the case for the vented box design. The resonant frequency is more or less unchanged.

So there are 3 main reasons to go with the vented box design.

1) We can extend the low frequency performance.

2) It allows us to better control the cone movement, thus allowing a much higher power output.

3) By utilising radiation from the rear of the driver we can improve the loudspeaker's efficiency.

Loudspeaker manufactures were well aware of these benefits, but at the time they didn't have the tools to design vented loudspeakers that worked reliably.

That was until Neville Thiele came around. His major contribution was recognising that the loudspeaker was basically acting like a high pass filter, and that it was possible to apply filter design techniques directly to the design of loudspeakers.

Shortly after Richard Small, one of Neville's students, extended Neville's work and in 1973 published a series of seminal papers in the AES which provided a 'fool proof' way of designing vented cabinet loudspeakers. This method is still used by manufactures today, and is what we will be focusing in this module.

An important aspect of their work was the to realise that the driver itself was an important design parameter in the design process. When we looked at the sealed box we found that the total compliance was the box compliance in series with the driver suspension. The speaker did change things, but it in terms of a slight misalignment, it is quite forgiving.

For a vented cabinet design, if you go slightly off the design target it all goes horribly wrong. It was partly for this reason that they proposed the Thiele-Small parameters for characterising loudspeaker drivers. They went on to set up framework where either: a) you work out what driver parameters you need for a particular box, or b) what box parameters you need for a particular set of driver parameters.

So before we start getting into the method itself lets look a little more closely at how a vented cabinet design actually works.

The sealed cabinet was an attempt to create an infinite baffle, thus getting rid of any interference from the rear of the driver. Whilst this seamed like a good idea at the time, it turns out that we can actually utilise this rear radiation to increase the level of the speaker, i.e. to make it more efficient at some frequencies.

The problem is that the rear of the driver is 180° out of phase with the front. If we were just added them together they would effectively cancel (this is why we added the infinite baffle). The trick is, that by adding a port we are able to invert the phase of this rear radiation, and then re-radiate it from the front. So for a listener in front the speaker it is as if the driver and the vent are in phase. This effect can provide a substantial bass boost at low frequencies.

The benefits of a vent are however limited to low frequencies. What is the issue at high frequencies? Well, the driver and the vent will act as two radiators, and if the wave length is short relative to their separation, we can get considerable variation in phase differences between the two at a listener position. Obviously, at low frequencies, when we have long wave lengths, the two radiators look like they are in the same place, and so radiate as one. Luckily, we will see that the vent does't radiate very well at high frequencies, so this issue is avoided. And anyway, we are quite happy with the mid frequency performance of our loudspeaker, its just the low frequencies that we want to give a helping hand. So all in all, the vent is a nice addition (when designed correctly).

So the physical construction of a vented enclosure is pretty simple. We take a sealed cabinet, put a hole in it, then put a tube in the hole to make a vent. Typically, the vent/hole is made to be circular (these are the easier to design, and have less losses than rectangular vents).

At the moment we are just crossing our fingers and hoping that the vent somehow gives as a 180° phase shift. Soon we will go through and derived a lumped parameter equivalent circuit and see that this is in fact the case. But first, lets think about what our 'lumped elements' will be.

These elements are shown in figure 8.20 and listen below.

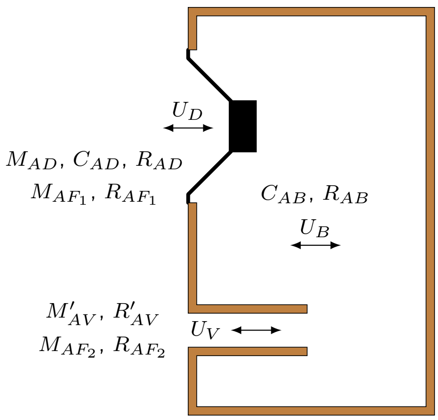


Figure 8.20: Vented enclosure.

- Driver:
  - M_{AD} - Mass of the driver in acoustic units
  - C_{AD} - Compliance of the driver in acoustic units
  - R_{AD} - Damping of the driver in acoustic units
  - M_{Af} - Inertial loading due to the radiation impedance
  - R_{Af} - Resistive loading due to the radiation impedance
  - U_D - Volume velocity of the driver
- Enclosure:
  - C_{AB} - Acoustic compliance of the cavity
  - R_{AB} - Acoustic damping within the cavity
  - U_B - Volume velocity within the cavity

- Vent:
  - M'_{AV} - Acoustic mass of the vent
  - R'_{AV} - Acoustic damping within the vent
  - M'_{Af} - Inertial loading on the vent due to the radiation impedance
  - R'_{Af} - Resistive loading on the vent due to the radiation impedance
  - U_V - Volume velocity of the vent mass

Most of our lumps are the same as in our sealed cabinet. We have the driver parameters: mechanical mass, suspension compliance and damping, radiation load (baffled piston), and the diaphragm volume velocity. Here we have given these all in acoustic domain units, hence the A subscript. The second subscript D tell us that these are properties of the driver in isolation. The radiation load includes inertial and resistive terms. These are usually included in the quoted parameters (S).

We have the cabinet parameters: acoustic compliance (dependant on the volume of the box) and damping (i.e. due to absorption in the box), but also we have a new volume velocity, corresponding to the air inside the box. Without the vent (i.e. for a sealed box), the box velocity is simply equal to the cone velocity.

For the vent we assume that the mass within it acts as if it were a solid lump. Then we have the following vent parameters: the mass of air in the vent, the resistive losses in the vent, the radiation load of the vent (pistonic, including both inertial and resistive components), and the vent volume velocity. We use the dash to denote that the mass and resistance do not include the air load.

The key difference between the vented and sealed design is the introduction of a second mass element, that of the vent. This means our vented loudspeaker is a 2 degree of freedom (DoF) system.

8.4.1 Two DoF Systems

The degree of freedom (DoF) of a mechanical (or acoustical) system is the number of independent parameters required to define its configuration. In general, when we consider dynamic systems (whether acoustical, mechanical or electrical), it is true that the number of resonant frequencies in the system response, is equal to the number of DoFs that the system possesses. This is a very general rule, and clearly applies to our loudspeaker.

With the addition of a vent mass, our loudspeaker has become a 2 DoF vibrating system (see figure 8.24). This means that our system should now exhibit two resonances.

8.4.2 Equivalent Circuits

Shown in figure 8.21 is a reminder of what our sealed box equivalent circuit looked like. As you can see, we have a single loop. Therefore we only have one current flowing. This means every component experiences the same current flow. Now since we used the impedance analogy, the current is analogous to volume velocity. So being in series means that the velocity is the same for every part.

Now that we have an additional vent mass, we have a 2 DoF system. Each mass has an associated volume velocity. So we no longer have a single volume velocity (i.e current) flowing. The total volume velocity in the system is supplied

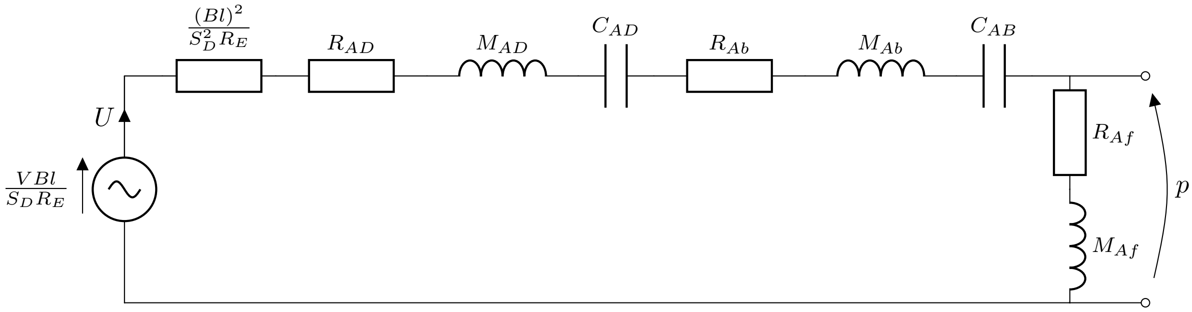


Figure 8.21: Complete equivalent circuit of a sealed cabinet loudspeaker including front and rear radiation loading.

by the driver. This is a fixed resource. Given that the volume velocity has to be conserved, we know that the sum of the enclosure and vent velocities must equal the volume velocity supplied by the driver, $U_D = U_V + U_B$. This is equivalent to saying that the box velocity is simple the relative velocity between the driver and vent,

$$U_B = U_D - U_V. \tag{8.71}$$

So now that we have two velocities, we have to have two loops, i.e. there has to be a branch that splits the driver volume velocity between the box and the vent. The resulting circuit is shown in figure 8.22. On one branch we have our sealed cabinet impedance. On the other we have the vent mass and resistance, and its radiation. In electrical terms, we can think of this additional vent branch as shorting out the enclosure, therefore reducing the velocity in the box.

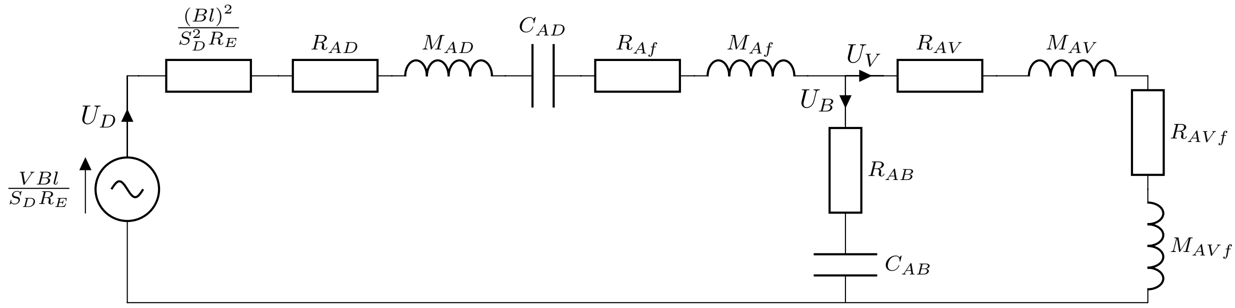


Figure 8.22: Complete equivalent circuit of a vented cabinet loudspeaker including front and rear radiation loading.

Now based on this equivalent circuit model, how do we chose the parameters such as box volume and port dimensions? Well before we get into that, lets look at how this new circuit resonates, and try to get some physical interpretation of what these resonances are doing.

Shown in figure 8.23 is the electrical impedance of a vented loudspeaker. Notice that we have two very well defined resonances. So how do we get this sort of electrical impedance? We apply an oscillating voltage to the driver whilst measuring the current being delivered. From this we can calculate the loudspeaker impedance. Then we just repeat this process at each frequency we are interested in.

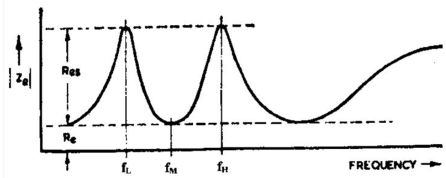


Figure 8.23: Electrical impedance of vented loudspeaker.

So what would you expect if we took the driver out of the enclosure and repeated this experiment? If you do this for a driver in free air you get a single peak, corresponding to the free air resonance of the driver. For a vented cabinet you get multiple peaks, which relate to the different modes of oscillation.

There are 3 important frequencies identified in this figure. The first lower peak f_L, the higher resonant peak f_H, and the minimum between them f_M. Also shown is the nominal electrical resistance, and the driver mechanical resistance in electrical units. Notice that if we increase the mechanical resistance, the peak value of the impedance also increases.

Now lets look at what is happening at each of these frequencies.

- *Lower resonance* f_L - At low frequencies the air in the cabinet does not compress, its acts as though it was rigid, i.e. the driver suspension is much softer than the cabinet of air. This means that the driver mass and the vent mass both bounce on the driver suspension. With reference to the mass-spring-mass in figure 8.24, this corresponds to the two masses moving in the same direction, i.e. in phase. This is just the same as having a single DoF system, with a total mass equal to the driver and vent mass together.

  One thing we need to remember here is, although we have drawn the velocity arrows in the same direction, physically, the driver and vent are moving in opposite directions (180° out of phase). As the driver moves out, it sucks the vent mass inwards, and visa versa. This means that we get very little radiated output from the loudspeaker. So when we design a vented cabinet, we want to end up with this lower resonant frequency below our working range.

  Note that the increased driver velocity, causes an increased back EMF, which is why we see an increase electrical impedance.

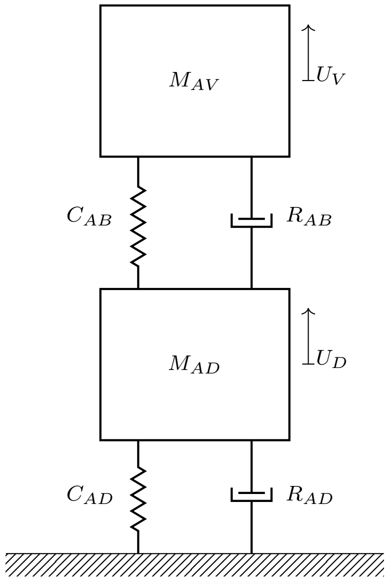


Figure 8.24: Lumped parameter mechanical model of a vented loudspeaker - relative motion at the first resonance f_L

- *Upper resonance* f_H - Around this resonance the driver suspension appears really soft, and so the driver and vent mass bounce together on the cavity compliance. With reference to the mass-spring-mass in figure 8.25, this corresponds to the two masses moving in opposite directions, i.e. out of phase. At first thought it might look as if these two will cancel out. We have to remember that U_D represents the velocity of the back of the driver. So what comes out the front is actually in phase. Therefore we had an efficient radiation, because the vent is helping out the driver.

  Note that the increased driver velocity results in increased back-emf and so the electrical impedance rises again.

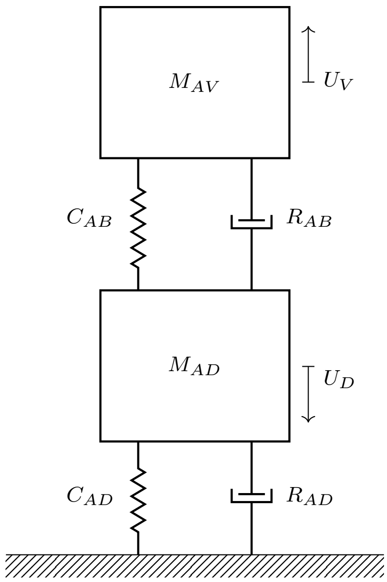


Figure 8.25: Lumped parameter mechanical model of a vented loudspeaker - relative motion at the upper resonance f_H

- *Minimum frequency* f_M - A minimum in the electrical impedance means that we have a minimum in the back EMF generated. With reference to the mass-spring-mass in figure 8.26, this corresponds to the driver mass hvaing a minimal velocity, i.e. it is as if the cone is fixed. Remember, the electrical and mechanical domains were coupled using the mobility analogue. So when the mechanical impedance is a maximum, the electrical impedance is a minimum. At maximum mechanical impedance the cone finds it very hard to move, so has a small velocity. This corresponds to a minimum in the electrical impedance, as we can see in figure 8.23.

  This is the same resonance frequency we would get is we were to rigidly fix the diaphragm so it couldn't move. What does this correspond to? The Helmholtz resonance of the box. And how do we calculate it? Easy, it only depends on the vent mass and the box compliance. Clearly, if this is the resonance when the driver is fixed, it must be independent of the driver. So it is purely a property of the box. We already know the resonant frequency of a Helmholtz resonator, for the vented cabinet it is given by,

$$f_M = \frac{1}{2\pi}\sqrt{\frac{1}{M_{AV}C_{AB}}} \tag{8.72}$$

where M_{AV} is the acoustic mass of the vent, and C_{AB} is the acoustic compliance of the cavity.

So what happens when we driver our loudspeaker at the Helmholtz frequency? Well, the driver barely moves, but the vent mass moves with a high velocity. This is the frequency that it wants to oscillate at! Because the vent is just a lump of air oscillating, it is very light, and so it couples well with the air outside. In turn we get lots of vent radiation and a high output pressure level.

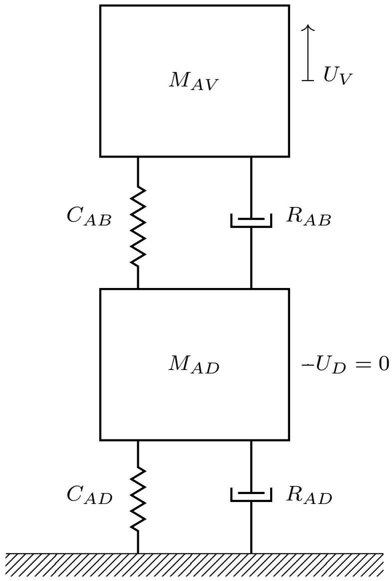


Figure 8.26: Lumped parameter mechanical model of a vented loudspeaker - relative motion at the minimum frequency f_M

Now what happens when we combine the driver radiated and vent radiated contributions? Well this is where the magic happens.

Around the lower resonance f_L the vent and driver mass move out of phase, and so we get destructive interference and very poor radiation. In fact, the combined effect of the vent and driver (which individually have a 12 dB/oct roll off) yield a 24 dB/oct low frequency roll off. Moving towards the minimum frequency f_M the driver's contribution tends to 0, whilst the vent radiation reaches a maximum. At this frequency, i.e. the Helmholtz resonance of the box, the vent is the primary source of radiation.

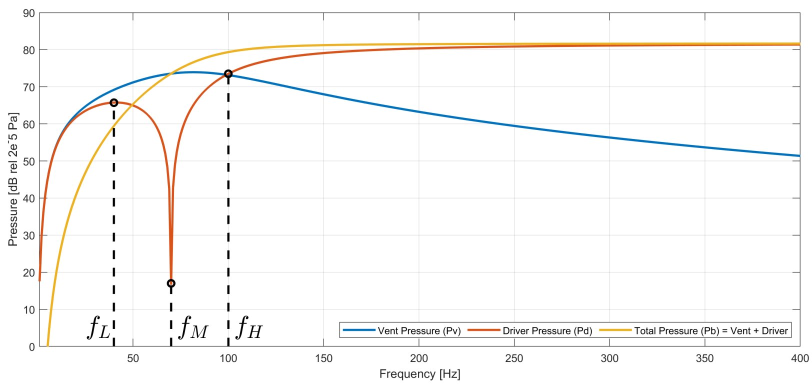


Figure 8.27: Pressure contributions of driver and vent.

As we increase the frequency further, the inertia of the vent mass starts to act like a low pass filter, and the vent radiation tails off. This is good though, we only want the vent to help us out at low frequencies. Between the Helmholtz and higher resonance f_H the vent and driver begin to radiate in phase, and so they interfere constructively. It is this added contribution that gives us a flat extended bass region. At high frequencies the driver will take over and the cabinet will look like a sealed box.

In contrast to a sealed cabinet design, the vented loudspeaker offers several advantages. Firstly, utilisation of the rear radiation has allowed us to extend the low frequency response of the loudspeaker to approx. the Helmholtz resonance f_M. The sealed cabinet however is limited by the driver resonance f_s. Secondly, we have improved the roll off rate, going from a 12 dB/oct to 24 dB/oct.

However, the above benefits come at the cost of a poorer transient response compared to the sealed cabinet design.

8.4.3 Equivalent Circuit Analysis

Having put together an equivalent circuit representing our vented loudspeaker we are now interested in analysing its performance.

To analyse the properties of figure 8.22 it is convenient to group terms together to form the complex impedances:

$$Z_D = \frac{(Bl)^2}{S_D^2 R_E} + R_{AD} + j\omega M_{AD} + \frac{1}{j\omega C_{AD}} + R_{Af} + j\omega M_{Af} \tag{8.73}$$

$$Z_B = R_{AB} + \frac{1}{j\omega C_{AB}} \tag{8.74}$$

$$Z_V = R_{AV} + j\omega M_{AV} + R_{AVf} + j\omega M_{AVf} \tag{8.75}$$

The corresponding equivalent circuit is shown in figure 8.28. From figure 8.28

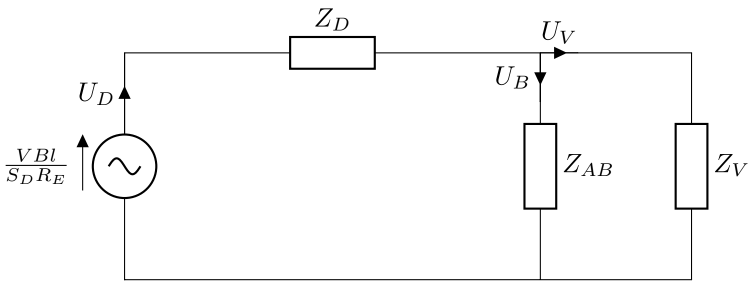


Figure 8.28: Complete equivalent circuit of a vented cabinet loudspeaker using grouped impedance elements.

the total impedance is given by,

$$Z_T = Z_D + \frac{Z_B Z_V}{Z_B + Z_V}. \tag{8.76}$$

Using the theory of AC current dividers the current (or volume velocity) through each branch can be determined as,

$$U_D = \frac{\frac{Bl}{S_D R_E}}{Z_T}, \tag{8.77}$$

$$U_B = \frac{Z_V}{Z_V + Z_B} U_D \tag{8.78}$$

and

$$U_V = \frac{Z_B}{Z_V + Z_B} U_D. \tag{8.79}$$

If we assume far field radiation at low frequencies then the path difference between the driver and the vent is negligible. In this case we can model the acoustic radiation as that of a monopole whose volume velocity is the sum of the vent and driver volume velocities.

There will be a contribution from the driver U_D, and an out of phase contribution from the vent, $-U_V$. So the total radiating volume velocity is $U_T = U_D + (-)U_V$. This is exactly what we determined for the box velocity. So finding the total radiating volume velocity, is equivalent to finding the box volume velocity. This is a really useful result.

Substituting the driver velocity into the box velocity we obtain,

$$U_B = \frac{Z_V}{Z_V + Z_B} \frac{\frac{VBl}{S_D R_E}}{\left(Z_D + \frac{Z_B Z_V}{Z_B + Z_V}\right)}. \tag{8.80}$$

It is straightforward to rearrange the above as,

$$U_B = \frac{\frac{VBl}{S_D R_E}}{Z_D \left(\frac{Z_B}{Z_V} + 1\right) + Z_B}. \tag{8.81}$$

Now lets substitute back in our grouped impedance terms. For simplicity we will ignore the cabinet and vent damping, and vent radiation effects. The driver's radiation impedance terms (also the electrical resistance) are included in the free space mass and damping terms $M_{AS} = M_{AD} + M_{Af}$ and $R_{AS} = \frac{(Bl)^2}{S_D^2 R_E} + R_{AD} + R_{Af}$. Substituting in these terms we obtain,

$$U_B = \frac{\frac{VBl}{S_D R_E}}{j\omega M_{AS} + \left(R_{AS} + \frac{1}{j\omega C_{AD}}\right)\left(\frac{1}{j\omega M_{AV} j\omega C_{AB}} + 1\right) + \frac{1}{j\omega C_{AB}}}. \tag{8.82}$$

As with the sealed cabinet analysis, we factor out a $j\omega M_{AS}$,

$$U_B = \frac{\frac{VBl}{S_D R_E}}{j\omega M_{AS}\left(1 + \frac{R_{AS}}{j\omega M_{AS}} + \frac{1}{(j\omega)^2 C_{AD} M_{AS}}\right)\left(\frac{1}{(j\omega)^2 C_{AB} M_{AV}} + 1\right) + \frac{1}{j\omega C_{AB}}} \tag{8.83}$$

before substituting in for the free driver Q-factor, $Q_{TS}/\omega_s = M_{AS}/R_{AS}$, resonance frequency $\omega_s^2 = 1/M_{AS}C_{AD}$, and the Helmholtz cavity resonance $\omega_B = 1/C_{AB}M_{AV}$,

$$U_B = \frac{VBl}{j\omega M_{AS} S_D R_E}\left[\frac{1}{\left(1 + \frac{1}{Q_{TS}}\frac{\omega_s}{j\omega} + \left(\frac{\omega_s}{j\omega}\right)^2\right)\left(\left(\frac{\omega_B}{j\omega}\right)^2 + 1\right) + \frac{1}{(j\omega)^2 M_{AS} C_{AB}}}\right]. \tag{8.84}$$

The acoustic mass $M_{AS} = 1/C_{AS}\omega_s^2$ can be substituted in to yield,

$$U_B = \frac{VBl}{j\omega M_{AS} S_D R_E}\left[\frac{1}{\left(1 + \frac{1}{Q_{TS}}\frac{\omega_s}{j\omega} + \left(\frac{\omega_s}{j\omega}\right)^2\right)\left(\left(\frac{\omega_B}{j\omega}\right)^2 + 1\right) + \left(\frac{\omega_s}{j\omega}\right)^2 \frac{C_{AS}}{C_{AB}}}\right] \tag{8.85}$$

where $\alpha = \frac{C_{AS}}{C_{AB}}$ is an important design parameter called the compliance ratio.

After expanding the bracketed terms,

$$U_B = \frac{VBl}{j\omega M_{AS} S_D R_E}\left[\frac{1}{1 + \frac{1}{Q_{TS}}\frac{\omega_s}{j\omega} + \left(\frac{\omega_s}{j\omega}\right)^2 + \left(\frac{\omega_B}{j\omega}\right)^2 + \frac{1}{Q_{TS}}\frac{\omega_s}{j\omega}\left(\frac{\omega_B}{j\omega}\right)^2 + \left(\frac{\omega_B}{j\omega}\right)^2\left(\frac{\omega_s}{j\omega}\right)^2 + \left(\frac{\omega_s}{j\omega}\right)^2 \frac{C_{AS}}{C_{AB}}}\right] \tag{8.86}$$

the above can be rearranged as so,

$$U_B = \frac{VBl}{j\omega M_{AS} S_D R_E}\left[\frac{1}{1 + \frac{1}{Q_{TS}}\frac{\omega_s}{j\omega} + \left(1 + \frac{C_{AS}}{C_{AB}}\right)\left(\frac{\omega_s}{j\omega}\right)^2 + \left(\frac{\omega_B}{j\omega}\right)^2 + \frac{1}{Q_{TS}}\frac{\omega_s}{j\omega}\left(\frac{\omega_B}{j\omega}\right)^2 + \left(\frac{\omega_B}{j\omega}\right)^2\left(\frac{\omega_s}{j\omega}\right)^2}\right]. \tag{8.87}$$

Equation 8.87 describes the box volume velocity (equivalent to the total volume velocity). Like the sealed cabinet's volume velocity (see equation 8.47), the above equation is made up of two terms. The first constitutes a low pass like term, and is identical to the equivalent term in equation 8.47. The second,

$$F(j\omega) = \frac{1}{1 + \frac{1}{Q_{TS}}\frac{\omega_s}{j\omega} + \left(1 + \frac{C_{AS}}{C_{AB}}\right)\left(\frac{\omega_s}{j\omega}\right)^2 + \left(\frac{\omega_B}{j\omega}\right)^2 + \frac{1}{Q_{TS}}\frac{\omega_s}{j\omega}\left(\frac{\omega_B}{j\omega}\right)^2 + \left(\frac{\omega_B}{j\omega}\right)^2\left(\frac{\omega_s}{j\omega}\right)^2} \tag{8.88}$$

constitutes a 4th order high pass filter term. It will turn out, similarly to the sealed cabinet, it is this term that governs the frequency response of the vented loudspeaker.

The shape of $F(j\omega)$ is more complex than that of $E(j\omega)$ (the frequency response of a sealed cabinet). It will depend not only on the driver properties and cabinet volume, but also the vent geometry. Given that $F(j\omega)$ is a 4th order filter, a greater number of possible response shapes are available.

Notice that we have the ratio of mechanical and acoustic cavity compliances. The compliance ratio, $\alpha = \frac{C_{AS}}{C_{AB}}$, is a really important parameter when designing vented cabinets. We can equally define it in terms of the cavity volume and the equivalent acoustic volume of the driver suspension. We will see how to use this term a little later on when we go through the design procedure.

So lets consider the radiated pressure of our vented loudspeaker by assuming a monopole like radiation,

$$p(r,\omega) = \frac{j\rho_0 ckU_B}{4\pi r}. \tag{8.89}$$

Substituting in our total volume velocity,

$$p(r,\omega) = \frac{j\rho_0\omega}{4\pi r}\frac{VBl}{j\omega M_{AS}S_D R_E}F(j\omega). \tag{8.90}$$

it is clear that the frequency dependence in the first order low pass term of our volume velocity cancels with the frequency dependence in the monopole radiation,

$$p(r,\omega) = \frac{\rho_0}{4\pi r}\frac{VBl}{M_{AS}S_D R_E}F(j\omega). \tag{8.91}$$

We are left with a collection of constants, followed by our 4th order high pass term. The constants are related to the efficiency or sensitivity of the loudspeaker, whilst the 4th order high pass describes its frequency response.

We have shown that the low frequency response of a vented loudspeaker is governed by the 4th order high pass filter term $F(j\omega)$. Now suppose we want a loudspeaker to exhibit a particular shaped response, for example a Butterworth. How do we get this?

Well, we know that the Q-factor, free driver resonance, Helmholtz resonance, and compliance ratio are all functions of the mechanical properties of the driver and the box geometry. So one option would be to play around with these parameters (making sure that we don't consider parameter values that are physically impossible) until we achieve the response shape that we are after. Given that there are quite a few variables, this is not a very practical approach.

There is a far easier second option, thanks to the work of Thiele and Small.

8.4.4 Choosing an Alignment

In a 1971 paper Small proposed a procedure for choosing the alignment for vented cabinet loudspeakers. The main design tool was a series of charts like the one in figure 8.29 (this one corresponds to a lossless cabinet, i.e. we haven't added any absorptive material).

This paper was essentially responsible for the popularity of the vented cabinet, and it makes it exceptionally easy to design a good quality loudspeaker.

When we designing sealed cabinets, we chose the system Q-factor and typically solve for the necessary box volume. Now things are more difficult. But, this chart handles all the difficult stuff for you. So there is quite a lot going on in the chart so lets go through it bit by bit.

So what are our design steps?

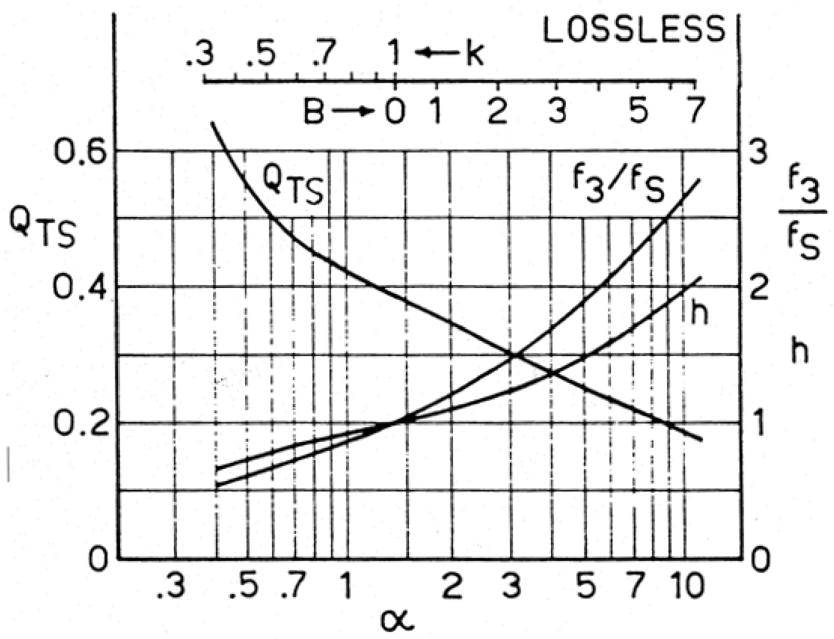


Figure 8.29: Small design chart for a lossless vented cabinet with a Butterworth frequency response.

- *Step 1:*

  1) First select a driver, this will give you the Q_{TS}, f_S and C_{AS} parameters of the driver.

- *Step 2:*

  1) Use the chart, follow the line labelled Q_{TS} and locate the Q_{TS} of your driver.

  2) Read off the α that corresponds to that Q_{TS} (remember, α (bottom scale) is the ratio of the driver compliance to the box compliance, so a large value of α implies a small box compliance compared with the driver compliance).

  3) Use the compliance ratio α to calculate box volume $V = C_{AB}\rho_0 c^2 = C_{AS}\rho_0 c^2/\alpha$. A speaker with small $Q_{TS} \to 0$ will yield a small $\alpha \to 0$ and so a small box (holy grail!).

- *Step 3:*

  1) For the same α, look up where it intersects with the other two lines.

  2) The parameter h is the ratio between Helmholtz and driver resonance, $h = f_h/f_s$.

  3) As the driver resonance is known, and the Helmholtz resonance is determined by the box compliance and the vent mass, h tells us the vent mass. From this we can determine the port dimensions. (Remember the port is an acoustic mass and so is proportional to length and inversely proportional to cross-section area. So it is tempting to make very narrow to increase mass, but this makes the particle velocity in vent large and generates turbulent flows. This is not good; it creates a chuffing sound!)

4) Small gives an empirical rule for port diameter (idea keep particle velocity lower than 5% speed of sound), $S_v \geq 0.8 f_b V_d$. It would be usual to account for flanged port at one end use with end corrections (we will cover this shortly).

5) Finally from the line f_3/f_s we can obtain f_3 which is the lower limit of the response (-3dB limit).

The example shown in figure 8.27 was calculated using the steps described above. It is important to recognise the importance of getting the a) correct loudspeaker Theile-Small parameters and b) determining the correct values of α and h from the Small chart in figure 8.29. Shown in figure 8.30 is the predicted response of a vented loudspeaker with the same cabinet geometry as in figure 8.27, but but with a different driver.

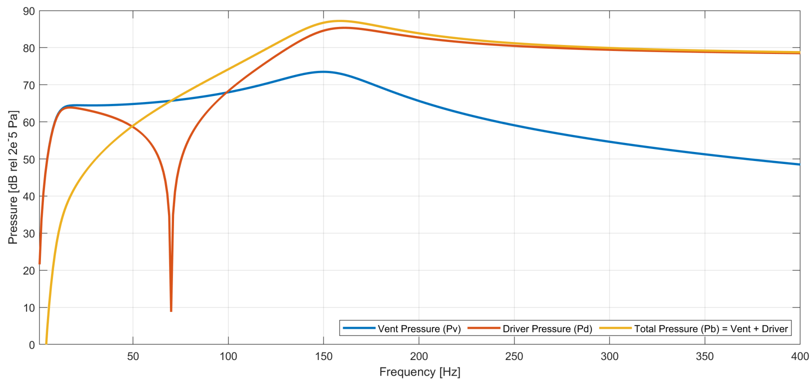


Figure 8.30: Example of misalignment due to incorrect driver parameters.

8.4.5 End Corrections

Suppose we have a mass of air, oscillating in a short vent. It is important to remember that, like the loudspeaker driver, there is an influence of the outside the vent. As the vent mass oscillates it must also move some amount of the air outside. This means that the *effective* length of the tube is longer than the physical length. We need to apply end corrections to any tube opening into an environment much larger than the tube itself.

Depending on the type of termination the type of end correction will vary. With vented cabinets in mind we are interested in either an infinite baffle termination (i.e. on the outside of the loudspeaker housing) or a free space termination (i.e. inside the cabinet).

Lets consider the infinite baffle termination first. Recall the radiation impedance of a rigid piston (see equation 7.20),

$$Z_{A,rad} \approx \frac{\rho_0 c k^2}{2\pi} + j\omega \frac{8\rho_0}{3\pi^2 a}. \tag{8.92}$$

This equation is also appropriate for modelling the oscillating vent mass so long as it only moves as a rigid body. We are only interested in the effect of added mass, and so we can ignore the resistive element of the above equation. Now recall the acoustic mass of a column of air is,

$$M_A = \frac{\rho_0 l}{S} = \frac{\rho_0 l}{\pi a^2} \tag{8.93}$$

where a is the vent radius and l is its *physical* length. Adding the radiation mass to the above the yields,

$$M_A = \frac{\rho_0 l}{\pi a^2} + \frac{8\rho_0}{3\pi^2 a}. \tag{8.94}$$

By multiplying the second term top and bottom by $a/3\pi$ we introduce a common denominator,

$$M_A = \frac{\rho_0 l}{\pi a^2} + \frac{8a\rho_0/3\pi}{\pi a^2} = \frac{\rho_0}{\pi a^2}\left(l + \frac{8}{3\pi}a\right). \tag{8.95}$$

From the above it is clear that by taking into account the radiation load of an infinite baffle the effective length of the vent becomes,

$$l' = l + \frac{8}{3\pi}a \approx l + 0.85a. \tag{8.96}$$

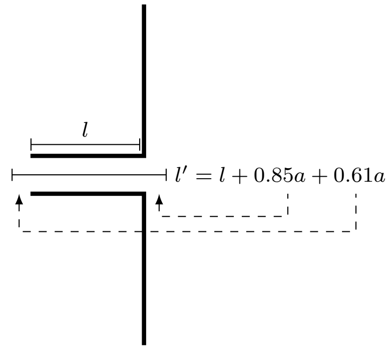


Figure 8.31: Effective length of the free/infinite baffle terminated vent.

Now lets consider the free space termination. This yields a slightly different radiation impedance given by,

$$Z_{A,rad} \approx \frac{0.0796\rho_0}{c} + j\omega\frac{0.1952\rho_0}{a}. \tag{8.97}$$

Adding the mass term from the above radiation impedance to the vent mass,

$$M_A = \frac{\rho_0 l}{\pi a^2} + \frac{0.1952\rho_0}{a} \tag{8.98}$$

which simplifies to,

$$M_A = \frac{\rho_0}{\pi a^2}\left(L + 0.195\rho_0\pi a\right) = \frac{\rho_0}{\pi a^2}\left(l + 0.6126a\right). \tag{8.99}$$

From the above it is clear that by taking into account the radiation load of free space the effective length of the vent becomes,

$$l' = l + 0.6126a. \tag{8.100}$$

A vent that is free at one end an terminated by an infinite baffle at the other is subject o both end corrections. Consequently, the total effective length of a loudspeaker vent is,

$$l' = l + \frac{8}{3\pi}a + 0.195\rho_0\pi a \approx l + (0.85 + 0.61)\,a. \tag{8.101}$$

Often when designing a vented loudspeaker the task is to find the appropriate vent dimensions to achieve a particular Helmholtz (cabinet) resonance. It is important to remember that the necessary dimensions include the effect of radiation load. Once found the *physical* vent length can then be determined from equation 8.101.

8.5 Performance Parameters

We have got as far as developing a model of our sealed enclosure loudspeaker, but what can we say about the performance of this loudspeaker? Well it turns out quite a bit.

First of all, what sort of performance parameters might we be interested in?

- *Frequency response* – describes the radiated sound pressure level as a function of frequency
- *Directivity* – describes the radiated sound pressure level as a function of angle.

- *Sensitivity* – describes the radiated sound pressure level in dB at a fixed distance of 1m for 1 watt of input power.
- *Efficiency* – is the percentage of electrical input power that is converted to acoustic power
- *Rated power* – how much power the speaker is designed to safely receive from an amplifier before it will start to distort

We have already had a look at the frequency response of our loudspeaker, and to some extent we have covered its directivity through our rigid piston model. But what about the others?

8.5.1 Directivity

We already had a little look at directivity when we covered our rigid piston model. You'll remember that at low frequencies our piston model behaved like a monopole, and radiated sound in all directions equally. At high frequencies the sound radiation becomes much more directional. If we go up high enough in frequency, the directivity develops lobes, where the main central lobe carries the most energy.

The directivity of a loudspeaker is a very important parameter when designing listening spaces. We want to maximise the radiation towards the listener, and minimise the radiation that just goes into the room. So how to we characterise the directivity?

Well there are two common ways of displaying it. The first is using a polar plot. This is were we display the angular dependence as a continuous curve, with a different plot for each frequency. The polar plot of a rigid piston (see figure 7.11) is reproduced in figure 8.32.

The other common approach is to plot the frequency dependence as a continuous curve, which a different plot for each angle, as in figure 8.33.

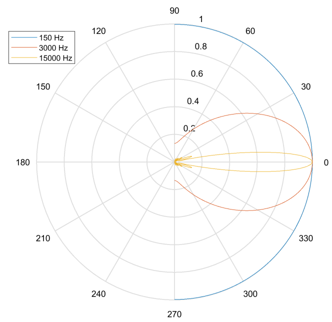


Figure 8.32: Polar response at 150, 3000 and 15000 Hz for a piston with $r = 0.15$ m.

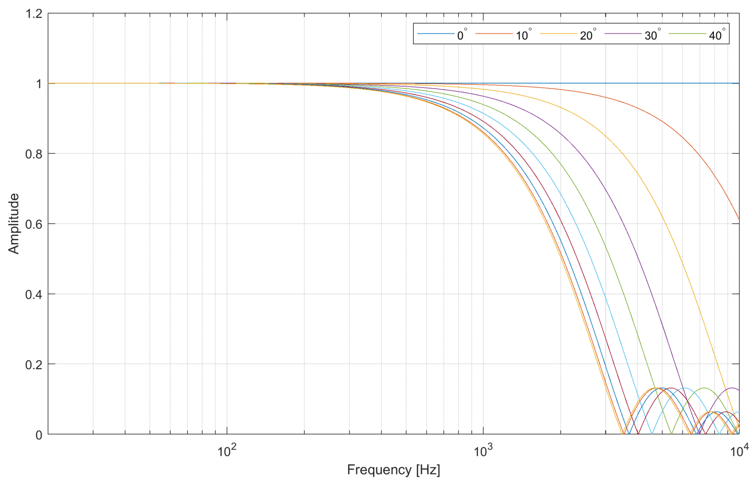


Figure 8.33: Waterfall plot of piston directivity at 10° intervals between 0° and 90°, for a piston with $r = 0.15$ m.

So these are graphical representations of the directivity. But what about some sort of numerical value that says how directive a source is? Well there are two common quantities for doing just this.

First, the directivity factor. This is the ratio of the sound intensity radiated by the loudspeaker (at a stated position and angle), to the sound intensity radiated by a point source (i.e. a monopole) whose acoustic power is the same as the loudspeaker,

$$Q(f) = \frac{I_r}{I_p}. \tag{8.102}$$

This is a frequency dependent parameter. At low frequencies, where the loudspeaker radiates as a monopole, the directivity factor is equal to 1. At higher frequencies, the intensity is typically focused forwards, and so I_r is greater than I_p and $Q(f)$ increases with frequency.

The issue with the directivity factor is that its value is often quite unwieldy. This is why we use the directivity index, which is 10 times the log of the directivity factor,

$$DI(f) = 10\log_{10} Q(f). \tag{8.103}$$

The directivity index is expressed in dB, and at low frequencies is equal to 0dB. When the radiated intensity is twice that of the equivalent monopole, the directivity index is equal to 3dB.

8.5.2 Acoustic Power and Intensity

Acoustic power is a scalar quantity that describes the amount of sound energy that is radiated per unit time. It is used routinely in industry to characterise how loud, for example domestic products are. It is defined in such a way that it is an independent property of the acoustic source (i.e. doesn't depend on the environment the source is put in). This is really useful because its lets us use the sound power to compare two sources. Acoustic power has the SI units of Watts, but is most often expressed as a dB value. So how do we calculate sound power?

Well first, we recall that mechanical power in general is defined as the dot product between force and velocity,

$$W = \mathbf{f} \cdot \mathbf{u}. \tag{8.104}$$

We know that pressure is related to force by area, so we can substitute this in.

$$W = Sp\hat{\mathbf{n}} \cdot \mathbf{u}. \tag{8.105}$$

But remember, force is a vector and pressure is a scalar. So we have to include the normal vector $\hat{n}$ to keep track of directions. If we consider the surface over which the pressure acts to be a hemisphere things get a bit easier ($S = 2\pi r^2$).

Now remember, the particle velocity is related to pressure by the characteristic acoustic impedance. Now substitute in for the pressure and impedance.

$$W = Sp\hat{\mathbf{n}} \cdot \frac{p}{Z_s}\hat{\mathbf{n}}. \tag{8.106}$$

We can now substitute in the radiated pressure from a piston (assuming low frequencies, i.e. $ka << 1$) and the specific acoustic impedance of a plane wave (we are assuming far field radiation).

$$W = 2\pi r^2 \left(\frac{\rho_0 ck}{2\pi r}\right)^2 \frac{1}{\rho_0 c^2}|U|^2 \tag{8.107}$$

After cancelling appropriate terms we end up with the sound power being equal to the real part of the acoustic front loading times the volume velocity squared,

$$W = \left(\frac{\rho_0 c k^2}{2\pi}\right)^2 |U|^2 = R_{Af}|U|^2. \tag{8.108}$$

Substituting in for the volume velocity we obtained,

$$W = R_{Af}\left|\frac{VBl}{\omega M_{AS} S_D R_E}\right|^2 |E(j\omega)|^2 = W_{ref}|E(j\omega)|^2, \tag{8.109}$$

where we define the reference power W_{ref} as,

$$W_{ref} = \frac{\rho_0}{2\pi c}\left|\frac{VBl}{M_{AS} S_D R_E}\right|^2. \tag{8.110}$$

There is another very useful quantity, closely related to acoustic power. Intensity is a vector quantity that describes the power carried by sound waves per unit area in a direction perpendicular to that area. Its definition is pressure times velocity,

$$I = p\mathbf{u}. \tag{8.111}$$

Now we can substitute velocity for pressure over impedance (don't forget the normal vector),

$$I = p\frac{p}{Z_s}\hat{\mathbf{n}}. \tag{8.112}$$

Now substitute in the pressure and impedance,

$$I = \left(\frac{\rho_0 c k}{2\pi r}\right)^2 \frac{1}{\rho_0 c^2}|U|^2\hat{\mathbf{n}}. \tag{8.113}$$

cancel the appropriate terms, and voila, we have the sound intensity being equal to the acoustic power divided by the hemisphere surface area,

$$I = \frac{W}{2\pi r^2} \tag{8.114}$$

But remember, intensity is a vector, so we still have our normal vector. For a hemisphere we know that this vector points radially outwards at all times.

We can see that as move further away from the source, the sound intensity gets smaller, whilst the acoustic power remains the same. This is why we use power to characterise acoustic sources, and not intensity.

8.5.3 Sensitivity/Efficiency

The sensitivity of a loudspeaker is one of the most important characteristics. Whilst the rated power of a loudspeaker tells us how much power a loudspeaker can handle, it does tell us what this power translated to as far as a dB level goes. For example, we might have a 1000 W loudspeaker, which produces 100 dB when driven at some particular voltage. Another loudspeaker, perhaps rated as 600 W could well produce a greater level, say 110 dB for the same input voltage.

The sensitivity of a loudspeaker to its input voltage is obviously a key parameters, perhaps more important than the rated power level. How is it defined?

It is the on axis radiated sound pressure level at a specified 1 meter (in an anechoic condition), when the loudspeaker is driven by 1 W of electrical input power.

$$Sens = 20\log_{10}\left(\frac{p}{p_{ref}}\right)_{\text{at 1W at 1m}} \tag{8.115}$$

To calculate the sensitivity then, we first need to figure out what voltage translates into 1 W of electrical power (this is done assuming a purely resistive impedance).

Electrical power is equal to voltage times current,

$$W = VI. \tag{8.116}$$

Substituting current for voltage over the nominal resistance gives power in terms of voltage squared and resistance,

$$W = \frac{V^2}{R_E}. \tag{8.117}$$

Rearranging this equation tells us that 1 W of power corresponds to a voltage equal to the square root of the resistance, $V = \sqrt{R_E}$.

Next, using our equivalent circuit model, with an input voltage of $\sqrt{R_E}$ we can predict the radiated sound pressure level, in dB at 1m. Lets look at a quick example.

Suppose we measure 0.2 Pa at 1 volt for an 8 Ohm speaker. What is the sensitivity? Well, to obtain 1 W of input power we need to driver to speaker with $\sqrt{R_E} = 2.83\ V$. Now remember, according to our equivalent circuit model the radiated pressure is directly proportional to the applied voltage. So if we measured 0.2 Pa under 1 V, we should measure $0.2 \times 2.83 = 0.57\ Pa$. This corresponds to a sensitivity of 89 dB.

Like the directivity, the sensitivity is frequency dependent. However, we often want a single value that encompasses all frequencies. But we have to remember that the ear is not equally sensitive to all frequencies, so we have to apply an appropriate weighting curve to the measured/predicted sound in order to reflect their aural effect. Because the frequency response of the ear is also sensitive to the level of sound, which weighting curve is used will depend on the amplitude level that is considered. For example, for a level of 55-85 phons a B weighting is used.

So what sort of sensitivity do typical loudspeakers have? Shown in figure 8.34 is a plot of the distribution of sensitivities across a range of loudspeakers. We can see that values around 85-90 are most typical. Speakers outside this range are usually either budget speakers, or very expensive professional monitors.

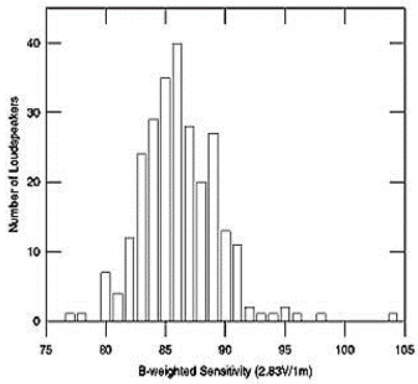


Figure 8.34: Distribution of sensitivities across a range of loudspeakers.

Loudspeaker efficiency is a similar parameter to sensitivity It tells us how efficient a loudspeaker is at converting electrical power into acoustic power, and is defined simply as their ratio (times a 100 to get it as a percentage value)

$$\eta = \frac{W_A}{W_E} \times 100\%. \tag{8.118}$$

Like the sensitivity, we assume the loudspeaker load is purely resistive, so we can easily calculate the electrical power as the voltage squared over the nominal resistance. Then our acoustic power is the product of the volume velocity and the acoustic radiation load.

$$\eta = \frac{R_{Af}|U|^2}{\frac{V^2}{R_E}} \times 100\% = \frac{R_{Af}R_E|U|^2}{V^2} \times 100\%. \tag{8.119}$$

This gives a nice straight forward for prediction the efficiency of our loudspeaker model.

It turns out that loudspeakers are incredibly inefficient. So it is not untypical to get efficiency ratings in the order of a couple percent. Shown in table 8.1 with some typical values of efficiency and their corresponding sensitivity.

Efficiency	Percent	Sensitivity
0.2	20 %	105 dB
0.1	10 %	102 dB
0.05	5 %	99 dB
0.02	2 %	95 dB
0.01	1 %	92 dB
0.005	0.5 %	89 dB
0.002	0.2 %	85 dB
0.001	0.1 %	82 dB

Table 8.1: Table of efficiency and corresponding sensitivity values.

Alternatively, given a particular efficiency the sensitivity can be calculated according to,

$$Sens = 112 + 10\log_{10}(\eta). \tag{8.120}$$

8.5.4 Rated Power

We have covered two parameters that describe the conversion of electrical power to acoustic power, the sensitivity and efficiency. Another important loudspeaker parameter is their rated power. This is the electrical power that a loudspeaker can handle before either the amount of distortion becomes unacceptable, or we have driver failure.

We haven't really covered distortion in this module, but it is important to be aware of it. Fundamentally what it means is that if we were to driver our loudspeaker with a single pure tone, the radiated pressure would consist of the pure tone, plus a series of harmonics (potentially other frequencies also) which were not present on the input signal. These harmonics are typically due to non-linearity in the system. For example, if we drive our loudspeaker with a high amplitude, the cone will displace quite far from its equilibrium. In doing this it will stretch the suspension. This will cause the compliance of the suspension to change. So now we have an amplitude dependent compliance. There are many other sources of non-linearity and hence distortion, but we wont covered those.

What about failure? The basic problem is that if you driver a speaker to hard you can permanently damage it or even cause it to fail. For example you might tear the suspension, burn out the voice coil, demagnetise the permanent magnet, etc.

To avoid these issues manufactures provide with their loudspeakers a power handling rating. This is the maximum input power the speaker should be driven by. Based on the power handling, and the loudspeaker sensitivity we can get an idea of the maximum sound pressure level our speaker can generate.

Understanding the two main causes for driver failure can help when it comes to understanding power ratings: they are thermal and mechanical.

- *Thermal failure* is the most common. It occurs when too much current is passed through the voice coil, and it is unable to dissipate the heat generated. We have seen that loudspeakers are incredibly inefficient! (in the region of 0.5-5%) The rest of the power supplied is turned into heat! So loudspeakers can get very hot. If it can dissipate the heat quick enough it will fail.

- *Mechanical failure* occurs when the cone, voice coil or suspension are forced beyond their limits. This is usually a result of the amplified peak voltage being to high. This causes an over excursion which results in the voice coil moving completely out of the voice coil, or 'bottoming out' and hitting the back plate of the loudspeaker. Poor design of the cabinet can further cause excursion problems.

So how do we rate the power handling of a loudspeaker? Well unfortunately, it's a little confusing because there are several different types of power ratings, and loudspeaker manufactures tend to mix and match which ones they present (often using which ever gives them the biggest number!).

The most well accepted power rating is the nominal power, which is specified according to an AES standard from 1984. It specifies that the driver should be tested in free air, orientated in the horizontal plane. The excitation signal used should be a band filtered pink noise extending 1 decade upward from the manufactures stated lowest usable frequency.

The power is then calculated as the squared voltage dived by the minimum driver impedance. The rated power is then the power the driver can withstand for 2 hours without permanent change in acoustical, mechanical or electrical characteristics greater than 10%.

Some other common power rating types are:

- The *RMS power* is the power obtained using the RMS voltage for its calculation.
- The *peak power* uses the peak voltage for its calculation. The peak power isn't particularly useful, as it typically cant be sustained for more than a second or so. Driving the loudspeaker continuously at the peak power would certainly cause damage.
- The *program power* is a term that derives from the old swept sine wave tests that were used for loudspeaker power. Although having no specific meaning nowadays, it's generally accepted that it is the amount of power that a speaker can handle during typical music or 'typical program' where frequency content and power constantly vary. For most manufactures it is simply 2 times the average power.

9 Magnetic Motor Design

So far we have talked in quite some detail about the mechanical and acoustical design of loudspeakers. But what about their electromagnetic design? At the heart of any modern moving coil loudspeaker is a permanent magnet. How do we design an optimum permanent magnet?

In the olden days, electromagnets were used instead, this was basically because of a lack of adequately strong permanent magnets. This electromagnet style of loudspeaker is known as a field coil loudspeaker, and are very rarely used these days.

So we already have a basic understanding of the general workings of the loudspeaker magnet. We have a voice coil that sits in magnetic field created by the permanent magnet. When a current is run through the voice coil we get a force that drivers the diaphragm upwards. When the current is revered, the force changes direction and drives the diaphragm downwards. And so we have a vibrating diaphragm.

What sort of permanent magnets can we use in loudspeaker design? It all depends on how much you want to pay. For example, we could use a rare earth magnet like neodymium (NdFeB, Neodymium Iron Boron). These magnets are terrifically strong, and so we only need small ones. This would save us considerable weight, but and the expense of cost. Alternatively, we can use a standard ferrite material (a ceramic compound that includes iron oxides). These are less powerful, but much cheaper. So typically, for the same sort of performance, lighter loudspeaker are more expensive, because stronger magnets are used (in a smaller quantity).

9.1 Basic Electromagnetism

So what is a magnetic field? We can think of it as a continuous field that permeates through space, and has the ability to exert a force on certain (magnetic) objects.

How do we create a magnetic field? Well there are two ways. You can generate a magnetic field by having electrical charge in motion. For example, the current running through a conductor (say a length of straight wire). Suppose some electrical charges (in the form of electrons) move left to right. This motion will create a magnetic field that circulates the wire according to the right hand rule (thumb in the direction of current flow). For a DC (constant flow) current, we get a constant magnetic field that wraps around conductor. For an AC current, the magnetic field oscillates, changing direction in time with the varying current.

The other way to generate a magnetic field is with permanent magnets. In a permanent magnet there is no current flow. However there are still charged particles are in motion within the material. Electrons have an orbit and a spin. These generate what are called electron magnetic dipole moments (the term magnetic moment really describes the magnetic strength and orientation or the magnet). You can almost think of a single atom as a tiny little bar magnet. Now when enough of these tiny bar magnets are aligned in the same direction, we create a permanent magnet.

9.1.1 Magnetic Fields

So how do we characterise the magnetic field generated by a moving charge or permanent magnet? We use what is called the magnetic strength H. Its units are amperes per meter, e.g. magnetic field strength of 1 ampere/m is generated at the centre of 1m diameter loop of conductor when 1 amp of current is applied.

Unfortunately, there is another closely related field which is also often termed the 'magnetic field'. This second field is denoted by B, and is often referred to as the magnetic flux density. We can think of the flux density as a measure of how closely packed together the field lines of a magnet are. Its important to understand that these two fields are fundamentally different. The quantity H denotes the strength of the field that is generated by the *magnet*. The quantity B denotes the response of the *medium* to that field. The two are related by what is called the magnetic permeability of the material.

9.1.1.1 Brief Flux Interlude

So this term 'magnetic flux', what is it? It quantifies the amount of a magnetic field flowing through a surface. It is important to remember that B is a vector field, so direction is important! The magnetic flux looks at the effect that the field has on an area. The angle that the field lines intersects this area is important. A field line passing through at a glancing angle will only contribute a small component of the field to the magnetic flux (because it includes only the component of the magnetic field vector which is normal to test area). A field line passing through at 90 deg (i.e. normal) will provide maximum flux.

For a uniform area A the magnetic flux can be expressed as,

$$\Phi = BA\cos\theta \tag{9.1}$$

where B is the flux density and θ is the angle at which the area is orientated see figure 9.1). The B field describes the density of this flux over the whole region. When $\theta = 0$, i.e. the B field is normal to the surface, $\Phi = BA$. When $\theta = 90°$ i.e. the B field is parallel to the surface, $\Phi = 0$.

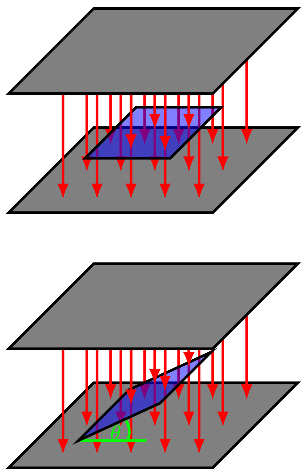


Figure 9.1: Magnetic flux

9.1.2 Magnetic Hysteresis

In free space, the flux density B and the field strength H are linearly related by the equation,

$$B = \mu_0 H \tag{9.2}$$

where $\mu_0 = 1.25663753 \times 10^{-6}$ is the permeability of free space. This tells us that the flux density (or how tightly packed the field lines are) increases proportionally with the strength of the field generated.

However, when looking at the flux density in a material things get a bit more complicated. In a material the relation between flux density and field strength can be non-linear.

For non-magnetic material we have nothing to worry about, the flux density and field strength remain linearly related, but via a different constant. Non-magnet materials are often specified by their relative permeability μ_r which when multiplied by the permeability of free space, give the materials magnetic permeability.

$$B = \mu_r \mu_0 H \tag{9.3}$$

For magnetic materials the flux density and field strength are related by a complex non-linear process, which we will cover shortly. But first we should introduce the different types of magnetism that exist, because there is more than one!

So in general there are 5 types of magnetic materials.

- *Diamagnetic and paramagnetic materials* only interact weakly with a magnetic field and the very weak effect disappears when the field disappears. These are typically considered non-magnetic materials.

- Magnetic materials can be either *Ferromagnetic or Ferrimagnetic*. Ferromagnetic materials are metals or metal alloys, like neodymium. In these materials the atoms are arranged in a lattice, and their magnetic moments can align parallel to each to generate a magnetic field. Ferrimagnetic materials are iron oxide based compounds, also known as ceramic magnets. Like Ferromagnetic materials, the magnetic moments align, but not all of them. As such, Ferrimagnetic materials are less strong than Ferromagnetic.

- Finally we have *antiferromagnetic* materials. These are like ferromagnetic but it turns out that for every moment pointing upwards, there is another one point downwards, and so the magnetic moments end up cancelling out.

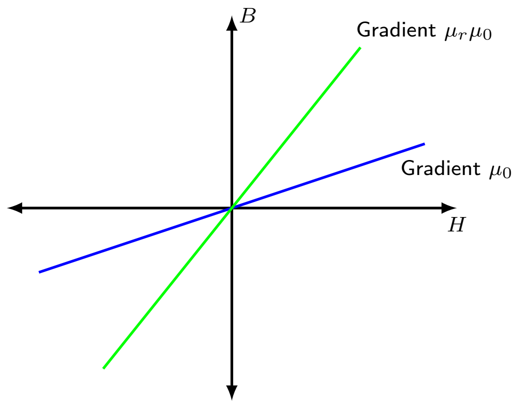


Figure 9.2: Relation between field strength and flux density for free space and non-magnetic materials.

So ferromagnetism is what we are interested in. But how does it work? We can use the theory of magnetic domains to explain how materials can be magnetised. Here is the idea.

Atoms create their own magnetic dipole moments. These dipole moments align over regions which we call domains. What creates a materials net magnetic field is the bulk effect of all of these domains acting together. When demagnetised these domains are orientated randomly across the material, and so the net effect is zero.

Now if we were to place the material in an externally generated magnetic field we can actually cause the domain walls to move. The domains favouring the external field are expanded, whilst those that oppose it are made smaller. Now the bulk effect is influenced more by the larger domains, and so the material is creating its own magnetic field (which is currently superimposed on top of the external one.)

For small movement of the domain walls this process of magnetisation is reversible if the applied field is reversed. For large movements however, the process can not be reversed by simply reversing the magnetic field. Extra work has to be done to move the walls back to a randomly orientated state.

It is this non-linear hysteresis process enables permanent magnets to exist. Hysteresis is the dependence of the state of a system on its history.

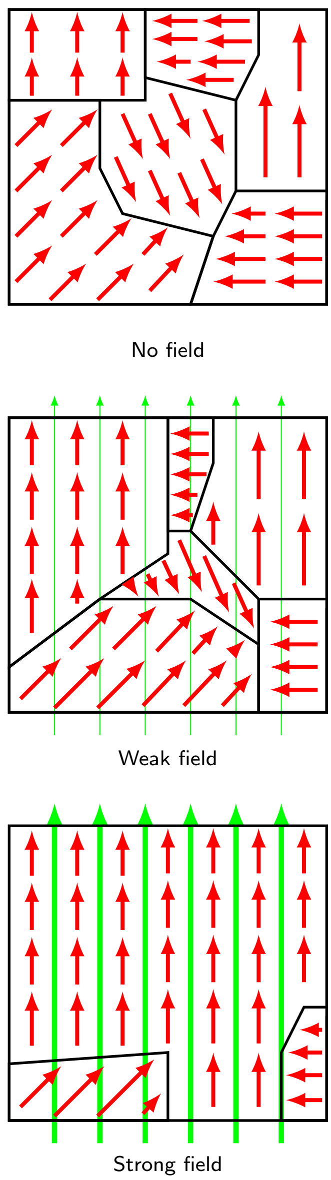


Figure 9.3: Domain extension during magnetisation

Shown in figure 9.4 is the hysteresis curve of a typical ferromagnetic material. It is a plot that represents the relation between an applied field strength H and the induced field or flux density B.

Now first thing, there are two curves here. The normal curve represents the total field, this includes the applied field and the field generated by the magnet itself. The intrinsic curve represents only the field generated by the magnet (i.e. does not include the applied field).

So lets follow the process of magnetisation. We start with our bulk zero magnetic field. Slowly we start to apply an external magnetic field, moving along the x axis. Following the intrinsic curve we see that the material slowly starts to generate its own magnet field. Eventually we reach a point where no matter the increase in the applied field, we can get no more out of the material (the B field reaches saturation). This is the saturation point of the material (will vary depending on the type of material).

Now we reduce the applied field, moving backwards along the x axis, until the applied field has been turned off, $H = 0$. The intrinsic field falls a little bit, but not to zero. We have no created a permanent magnet.

Now lets start reversing the applied field, going along the negative x axis. The first interesting point we reach is where the total field (i.e. the normal curve - the combined applied and generated field) equals 0. This means the applied field (which is now opposite in direction) cancels exactly with the field generated by the material. This point is known as the coercive force H_C. But if we look at the intrinsic field, we can see that the material is still quite strongly magnetised.

So we continue to increase the strength of the applied field (in the opposite direction still). Eventually we reach a point where the applied field is strong enough to demagnetise the material, i.e. the intrinsic curve has reached 0. This point is called the intrinsic coercivity, H_{Ci}.

If we continue to increase the applied fields strength we will end up creating a permanent magnet with an opposite polarity as we had before. This procedure then repeats forwards and backwards, giving up this continuous hysteresis curve.

For loudspeakers we want to look at the intrinsic curve. In particular it is the flat region above H_C that we are interested in. In this region we get a relatively constant flux density from the magnet. We need B to be constant in out loudspeaker model, as this is what enables us to model the force as a linear function of the applied current. This means that we need to make sure that the field generated by our voice coil H is within these limits. If the voice coil field is too large then you can accidentality demagnetise your loudspeaker. That said, the more likely cause of demagnetisation is due to the heat generated by operating the loudspeaker at high levels.

So it is this second quadrant that is most important for use. It is called the demagnetisation curve, and it tells us about the resilience of a magnet to an external field.

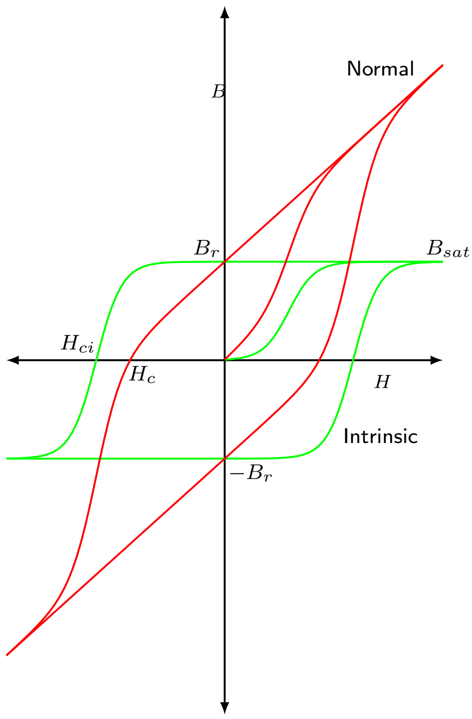


Figure 9.4: Hysteresis curve for ferromagnetic material.

9.1.3 Lorentz Force

Before we look at the structural design of magnets for loudspeakers, lets look a little more closely at how they drive our voice coil. We are interested in the passage of electrons (i.e. a current) through the voice coil (i.e. a wire), and the effect of the magnetic field B. To understand this we have to introduce the

Lorentz force law.

So what does this force law say? It tells how a magnetic field influences a charged particle in motion. It states that force on a particle of charge q is proportional to the electric field $\vec{E}$ present, and the cross product of the particle velocity $\vec{v}$ and magnetic field $\vec{B}$.

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{9.4}$$

The direction of this force can be determined using the right hand rule. If your index finger points in the direction of the charge direction, your middle finger in the direction of the magnetic field, your thumb will indicate the direction of the resulting force, ass illustrated in figure 9.6.

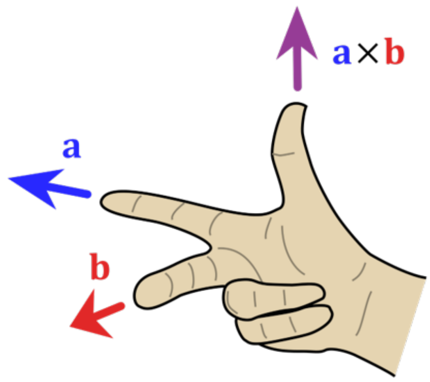


Figure 9.5: Right hand rule.

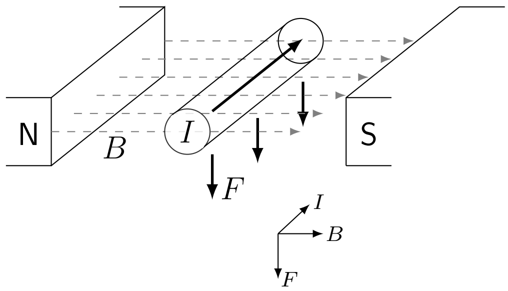


Figure 9.6: Lorentz force on a charge carrying conductor.

Now, recall that velocity is the distance travelled over the time taken,

$$\vec{v} = \frac{\Delta \vec{x}}{\Delta t}. \tag{9.5}$$

Lets substitute this into the Lorentz force law (ignoring the electric field term because we are considering the effect of a permanent magnet).

$$\vec{F} = q\left(\frac{\Delta \vec{x}}{\Delta t} \times \vec{B}\right) = \frac{q}{\Delta t}\left(\Delta \vec{x} \times \vec{B}\right) \tag{9.6}$$

Now recall that current is defined as the rate of flow of charge,

$$i = \frac{q}{\Delta t} \tag{9.7}$$

which when substituted into the above yields,

$$\vec{F} = i\left(\Delta \vec{x} \times \vec{B}\right). \tag{9.8}$$

Equation 9.8 states that the force $\vec{F}$ is proportional to current, and is in the direction of particle displacement cross the field direction. To simplify this equation we can rewrite the cross product in term of the vector magnitudes and the angle θ between them,

$$\Delta \vec{x} \times \vec{B} = |\Delta \vec{x}||\vec{B}| \sin\theta \hat{n} \tag{9.9}$$

where $\hat{n}$ is a unit vector in the direction perpendicular to $\Delta \vec{x}$ and $\vec{B}$. Noting that the total distance travelled by charged particle is the length of the voice coil, the above may be rewritten as so,

$$\vec{F} = i|\Delta \vec{x}||\vec{B}| \sin\theta \hat{n} \rightarrow Bli. \tag{9.10}$$

This is exactly the linear force equation that been assuming thus far. We have now shown that it is simply a consequence of the Lorentz force law.

9.2 Magnet Structure

The general idea in magnet design, is that we can use metal components (typically steel) to help direct the magnetic field lines, concentrating them where necessary (i.e. in the voice coil gap).

For practical reasons voice coils are cylindrical, so typical magnets are toroidal. The magnetic field lines (flux) of a toroidal magnet prefers the outside path. This is because like fluxes repel each other (much like the north end of two magnets repel). Shown in figure 9.7 the field lines of a stand-alone toroidal magnet.

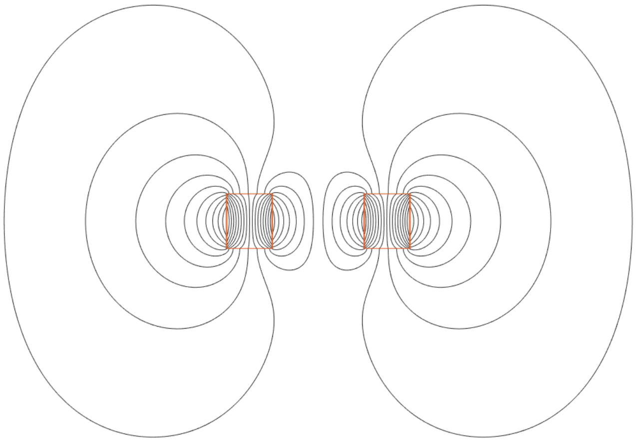


Figure 9.7: Stand-alone toroidal magnet with field lines. Flux tends to take path outside the ring magnet.

By including a pole pieces (i.e. a structure through which the magnetic flux prefers to travel) we can provide the flux with a much easier path. We can think of the magnetic flux as analogous to current; it prefers the path of least resistance. The magnetic permeability of steel, for example, is much much greater than air, so the flux would pass mostly through a steel pole piece. Shown in figure 9.8 is an example of a pole piece design for the same toroidal magnet. Using this

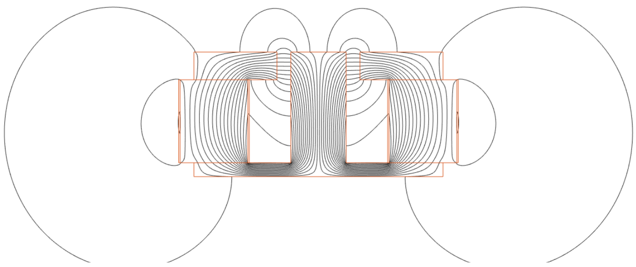


Figure 9.8: Toroidal magnet with central pole piece to redirect the field lines. high leakage of conventional exterior magnet.

design we can concentrate the flux density over the small gap between the pole piece and the other end of the magnet. This gives us a much greater force when applied over a voice coil (which fits within this small gap).

There will however, always be some amount of leakage, i.e. some flux always decides to travel around the outside of the magnet. The aim of magnet design is to minimise this flux leakage, and thus maximise the useful flux across the gap. With this in mind we may wish to improve the pole piece design for our ring magnet (perhaps one that follows the natural flux path?). Figure 9.9 offers such a design. We still have the same ring magnet, but we have the pole piece extend

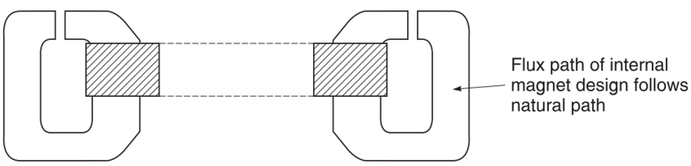


Figure 9.9: Toroidal magnet with external pole piece to redirect the field lines according to their natural path.

around the outside. This requires that the voice coil is kept outside the magnet (which means we can have a longer coil). This new pole piece arrangement also follows the natural flux path, and so reduces the leakage even further. Shown in figure 9.10 is a close up of the field lines through this external pole piece.

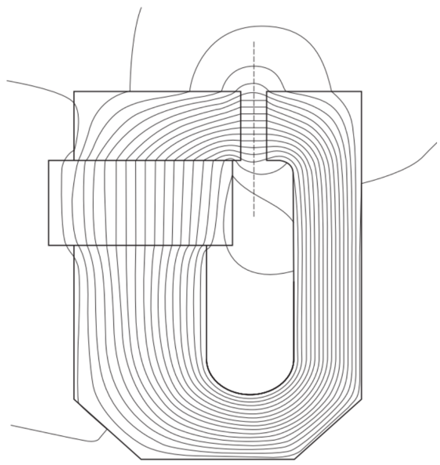


Figure 9.10: Field lines through a toroidal magnet with external pole piece.

It is clear that the field lines follow tightly the pole piece shape, and concentrate across the voice coil gap. Notice however, that the field lines are not all parallel across this gap. We have some fringing around the top and bottom. We need to be careful about this, because it will introduce non-linearities.

One of the most important requirements of a magnet's design is that it achieves a near constant flux density across the gap. This is because we want the voice coil to receiver the same amount of forcing irrespective of its displacement. Say if the flux were much less near the top and bottom of the voice coil gap, then when the driver is displaced sufficiently far, the force exerted on the voice coil will be weaker, due to the reduced magnetic flux. What effect does this have? It introduces a non-linear force model, where the force depends on the displacement of the driver. Not only would this hugely complicate our equivalent circuit model, but it would also introduce subjectively unwanted artefacts. Imagine driving the loudspeaker with a pure sine wave, as the peaks of the sinusoid reach maximum extension of the voice coil, they are clipped due to the reduced motion. This is distortion. These are complications that we want to avoid.

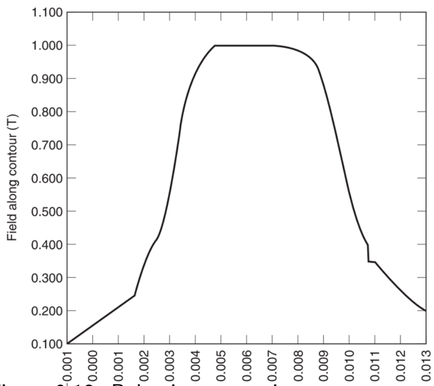


Figure 9.11: Distribution of flux density across the voice coil gap.

So what are we trying to achieve with our magnet design? A uniform flux distribution across the entire gap. If we were to plot the flux density as a function of distance along the voice coil gap, we want to maximise this flat region. Shown in figure 9.11 is the flux distribution across the external pole piece gap in figure 9.10. Notice the flat top. This is the region in which we want to operate our loudspeaker. Across this region the voice coil receives the same flux density. What

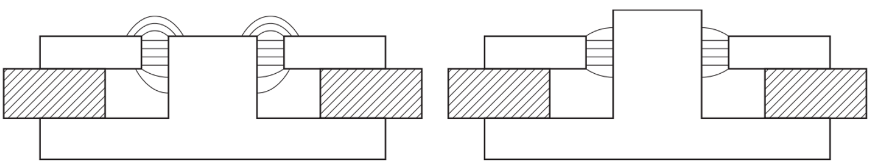


Figure 9.12: Pole piece extension

can we do to flatten, or smooth out the flux density across the gap? Two options. Number 1, we can re design the magnet pole piece so that it extends beyond the gap. This would reduce the fringe flux at the top of the gap, as in figure 9.12.

Alternatively, we can play around with the voice coil sizing.

9.3 Voice Coil Design

the voice coil is the part of the loudspeaker where the driving force (i.e. the Lorentz force) develops. We want this driving force to be linear (i.e. not to depend on the displacement of the voice coil). If the voice coil moves too far the flux density will decrease and non-linearities will occur.

For a tweeter (high frequency driver) the movement of the voice coil is very small. Since the flux naturally extends a little beyond the gap the voice coil can be made the same size as the gap without introducing any non-linearities.

For a woofer we require a much larger displacement to achieve the level wanted. So to ensure a constant flux the coil has to be made either longer or shorter than the gap.

If the coil is made longer than the gap the main passage of flux will remain within the area of the voice coil (to some extent). This called an *overhung coil*. The downside of this design is that it is less efficient, as the part of the voice coil that sits outside of the flux isn't really doing anything. Its just extra baggage. Although the heavier coil reduces the sensitivity of the driver, if the coil does exceed its limit, the non-linearity is 'soft', i.e. has a slow on set.

If the coil is made shorter than the gap as it is displaced in remains entirely within the uniform flux region. This type of design is called an *underhung coil*. The downside of this design is that the smaller coil has less windings and so generates a smaller force. As such it requires a larger magnet structure and is therefore heavy and more expensive. If the coil exceeds its limits, we get a hard non-linearity, i.e. it has a quick onset.

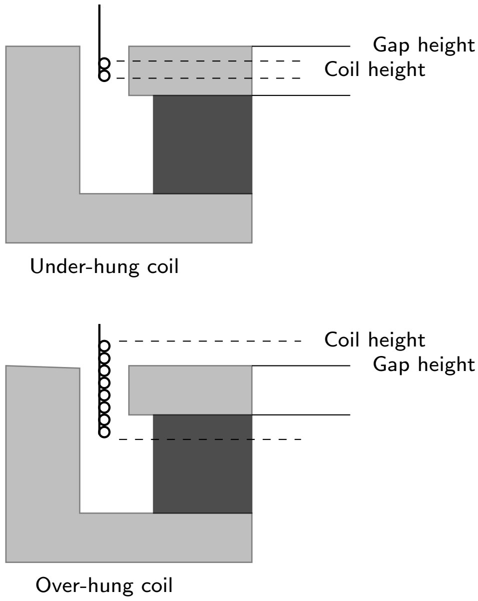


Figure 9.13: Voice coil sizing for a woofer design.

9.4 Magnetic Circuit Design

Like we did for the mechanical and acoustic domain, it is possible to formulate an equivalent circuit analogy for magnetic systems. To do so we make the following equivalences,

$$MMF \leftrightarrow V \tag{9.11}$$

$$\Phi \leftrightarrow i \tag{9.12}$$

$$R \leftrightarrow R \tag{9.13}$$

where the magneto-motive force MMF is equivalent to voltage V, magnetic flux Φ is equivalent to electrical current i, and magnetic reluctance R is equivalent to electrical resistance R.

These quantities are related through Hopkin's law, an equivalent of Ohm's law for magnetic circuits,

$$\Phi = \frac{MMF}{R}. \tag{9.14}$$

Note that although conceptually we may interpret the MMF as a 'force' that 'drives' the flux through a magnetic circuit (this is an analogy after all), it is important to remember that it is not a true force since flux doesn't flow, it is a fixed field.

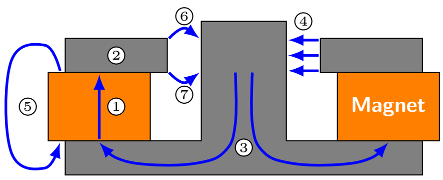


Figure 9.14: Transmission line enclosure.

An obstruction in a magnetic circuit to the magnetic flux is called reluctance. In magnetic circuit design is more typical to deal with the reciprocal of reluctance, the so called permanence,

$$P = \frac{1}{R}. \tag{9.15}$$

The magnetic permanence of a material is determined by its geometry and relative permeability as so,

$$P = \frac{\mu_r \mu_0 A}{L}. \tag{9.16}$$

where μ_r is the relative permeability of the material, A is its cross sectional area, and L is its length.

Consider the typical toroidal loudspeaker magnet design in figure 9.14. The labelled elements correspond to:

1 Magnet MMF

2 Top plate reluctance

3 Centre pole reluctance

4 Gap reluctance

5 Magnet perimeter leakage

6 Gap fringing flux

7 Gap fringing flux

Shown in figure 9.15 is the equivalent magnetic circuit

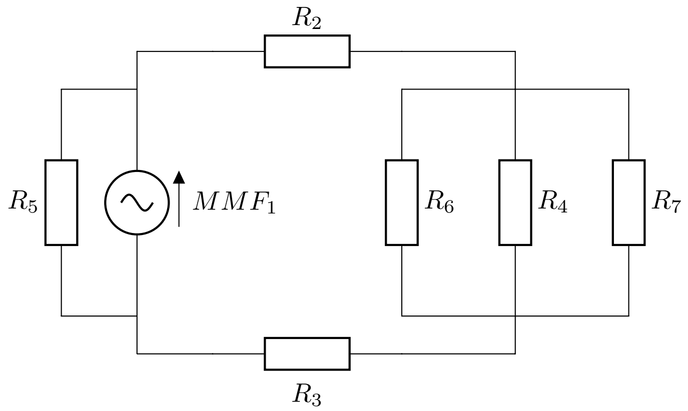


Figure 9.15: Equivalent magnetic circuit for toroidal loudspeaker.

Neglecting the pole piece permanence and any parallel flux paths, the total permanence of the magnet structure is given by,

$$P_T = P_f + P_g \tag{9.17}$$

where P_f, and P_g are the permanences of the fringe paths and the air gap.

The magneto-motive force across the magnet element is given by the product of its length and field strength,

$$MMF_m = H_m L_m. \tag{9.18}$$

Similarly for the air gap,

$$MMF_g = -H_g L_g. \tag{9.19}$$

Note that the air gap MMF is an opposing one, unlike the magnet element.

For now let us assume that there are no fringe flux paths, and that the flux is contained entirely within the magnet, pole piece and air gap. In this case, according to Kirchhoff's 1st law (conservation of current) the flux through the magnet element must be equal to the flux through the air gap,

$$\Phi_m = \Phi_g. \tag{9.20}$$

The flux may readily be expressed in terms of a flux density and corresponding area,

$$B_m A_m = B_g A_g. \tag{9.21}$$

Under the assumption the pole piece/fringing path MMFs are negligible, according to Kirchhoff's 2nd law (conservation of energy) the magnet and gap MMFs must sum to 0,

$$MMF_m + MMF_g = 0. \tag{9.22}$$

From equations 9.18 and 9.19 we have,

$$H_m L_m = H_g L_g \tag{9.23}$$

which, recalling that $H_g = \frac{B_g}{\mu_0}$ can be rearranged as,

$$H_g = \frac{H_m L_m}{L_g} = \frac{B_g}{\mu_0}. \tag{9.24}$$

From equation 9.21 we have,

$$B_g = \frac{B_m A_m}{A_g} \tag{9.25}$$

which can be substituted into equation 9.24 to yield,

$$\mu_0 \frac{H_m L_m}{L_g} = \frac{B_m A_m}{A_g}. \tag{9.26}$$

From the above we can obtain the following equation,

$$\frac{B_m}{\mu_0 H_m} = \frac{L_m A_g}{L_g A_m}. \tag{9.27}$$

Equation 9.27 is the load line of the magnetic circuit. The intersection of this line with the demagnetization curve represents the operating point of the magnet. In terms of the demagnetization curve the flux density has decreased from B_r, to B_m and a negative potential $L_m H_m$ has developed which is equal to the potential drop in the air gap $L_g H_g$.

Note that equation 9.27 was derived under the assumption that a) all the flux is contained within the magnet, pole piece and air gap, and that b) that the pole piece/fringing MMFs were negligible.

To account for these we begin by defining the leakage flux factor,

$$\sigma_\Phi = \frac{\phi_m}{\phi_g} > 1 \tag{9.28}$$

where σ_Φ represents the proportion of flux that is lost (e.g. through fringing paths). From the above we have,

$$\Phi_m = \sigma_\Phi \Phi_g \tag{9.29}$$

Similarly, we can define the MMF loss factor,

$$\sigma_{MMF} = \frac{MMF_m}{MMF_g} > 1 \tag{9.30}$$

where σ_{MMF} represents the proportion of MMF that is unaccounted for (e.g. through pole pieces or fringing paths). From the above equation 9.22 becomes,

$$MMF_m + (-)\sigma_{MMF} MMF_g = 0. \tag{9.31}$$

Equations 9.31 and 9.29 lead to the load line equation,

$$\frac{B_m}{\mu_0 H_m} = -\frac{\sigma_\Phi}{\sigma_{MMF}} \frac{L_m A_g}{L_g A_m} \tag{9.32}$$

or

$$B_m = -\frac{\sigma_\Phi}{\sigma_{MMF}} \frac{L_m A_g}{L_g A_m} \mu_0 H_m. \tag{9.33}$$

The load line equation describes a negative slope on the magnet demagnetization curve. The right hand side term $\frac{\sigma_\Phi}{\sigma_{MMF}}\frac{L_m A_g}{L_g A_m}$ is often called the permeance coefficient P_C.

Usually the designer will select the dimensions so as to operate at given B_m/H_m ratio. Often it is the point BH_{max} where the product B_m H_m is at a maximum and consequently the magnet's volume will be a minimum. From equation 9.21 and 9.23, it is simple to show that,

$$V_m = L_m A_m = \frac{B_g A_g H_g L_g}{B_m H_m} \tag{9.34}$$

where V_m is the volume of the magnet. Clearly the maximum value of $B_m H_m$ corresponds to the minimum volume V_m.

The gradient of the load line, i.e. equation 9.32, is called the permanence coefficient. It relates the magnet's field strength and flux density for a given magnet design and gap geometry. Increasing the gap area, or decreasing the gap length will increase the permanence coefficient, moving the working point up the normal curve. In the limit that the gap length tends to zero the gradient will tend to infinity, i.e. a vertical line. In this case we have a closed magnetic circuit, i.e. there are no negative potentials, and magnet's flux density corresponds to the remanent flux density B_r.

9.4.1 Load-line

Equation 9.33 is the key result of this chapter. It describes how the B and H fields in a magnetic circuit are related to one another given that there is an air gap present. To improve the efficiency of a loudspeaker driver we want to maximize the flux density B_m (this is what drives the loudspeaker, via the Lorentz force) given a field strength H_m (this is a property of the magnet). The presence of an air gap acts as an obstruction to the 'flow of flux', causing a decrease in its density. The intersection of the load-line and the normal-curve defines the operating point of the magnet (see figure 9.16). The larger the air gap or the smaller the magnet, the further down the normal curve the operating point moves. The position of the working point is critical to ensure that the H field is not increased to a point where the field could exceed H_{ci} and the magnet will become permanently demagnetised.

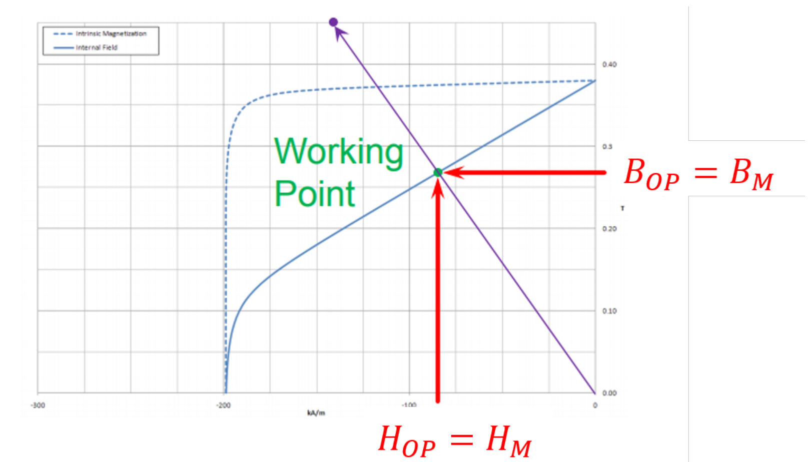


Figure 9.16: Example of a load line and its intersection with the normal curve, i.e. the operating point.

9.4.2 External Magnetic Fields - Operating Loudspeaker

In its passive state (i.e. with no external current applied) the loudspeaker magnet is operating at the working point, $-H_{op}$. This corresponds to the negative potential introduced by the air gap. The resulting flux density is given by B_{op}.

Whilst operating an alternating current runs through the loudspeaker's voice coil. This current interacts with the permanent magnetic field via the Lorentz force, causing a periodic displacement of the cone. However, if we run a current through the voice coil we also introduce an additional *induced* magnetic field, H_{ext}. This external field has the effect of shifting the load-line, and with it the working point of the magnet. See for example figure 9.17, where an additional field ΔH is applied. For small currents, the working point will oscillate forwards

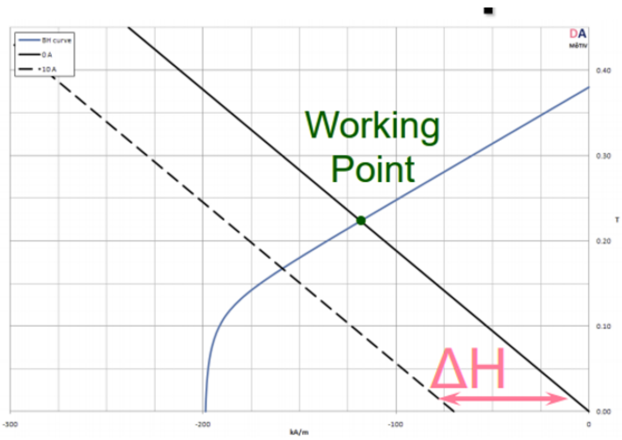


Figure 9.17: Example of an external field shifting the load-line and working point.

and backwards, remaining at all times on the normal curve. Clearly, if the applied current is too great, the additional field will shift the working point over the knee of the normal curve. Now when the external magnetic field is reversed, rather than the working point following the normal curve it follows a new path parallel to, but below, the normal curve. This is called the recoil path. At this point the loudspeaker has suffered some permanent demagnetisation.

We can use normal curve knee to specify the maximum external field strength, and therefore the maximum permissible current.

9.4.3 Temperature Dependence

Another really important feature of a permanent magnet is its temperature dependence. The normal curve of a magnet can be strongly dependent on temperate and material. Shown in figure 9.18 is an example of Neodymium.

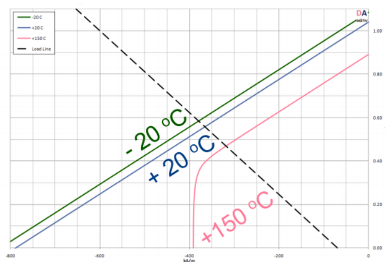


Figure 9.18: Normal curve temperature dependence of Neodymium.

Between +/- 20 degrees the normal curve has a broad flat slope. This is good as it means we can avoid demagnetisation around room temperature. At 150 degrees however, the knee of the normal curve occurs much earlier. Now we run the serious risk of demagnetisation if the speaker is driver too hard. This is particularly important for loudspeaker, because they can get very hot! Remember, they are terribly inefficient (less than 5%!). This energy has to go somewhere (heat!). Also notice that as we increase temperature, the normal curve drops in level, i.e. the sensitivity is reduced.

Shown in figure 9.19 is a ceramic ferrite example. Again, the load line is varying greatly with temperature. Again, we see that by increasing the temperature the normal line drops. Interestingly however, unlike neodymium which became less linear at high temperature, this ceramic magnet gets more linear at high temperatures. In this case our loudspeaker could operate safely at high temperatures, but at low temperatures we run the risk of demagnetisation!

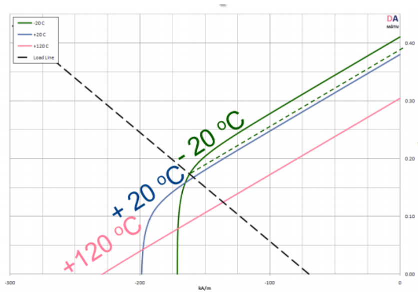


Figure 9.19: Normal curve temperature dependence of Neodymium.

9.4.4 Magnet Performance

Shown in figure 9.20 is a typical data sheet supplied with a magnet used for a loudspeaker. Both the normal and intrinsic curves are shown for a number of temperatures. Also shown are a series of load-lines. Lets use this chart to assess the performance of its associated magnet.

To start let us suppose we have calculated the permeance coefficient for a particular gap geometry. This may be done using the load-line equation we derived from our magnetic circuit, or some more complex model (e.g. a Finite Element model). Say the value obtained is 0.6. To find the operating point of the magnet we follow the 0.6 load-line to where it intersects the normal curve. Considering the 20° normal curve, this occurs at approx $H_{op} = -7.5$, at which $B_{op} = 4.5$. We can see that in this region the normal curve is in its linear regime. For small applied fields the working point will move forwards and backwards, remaining on the normal curve.

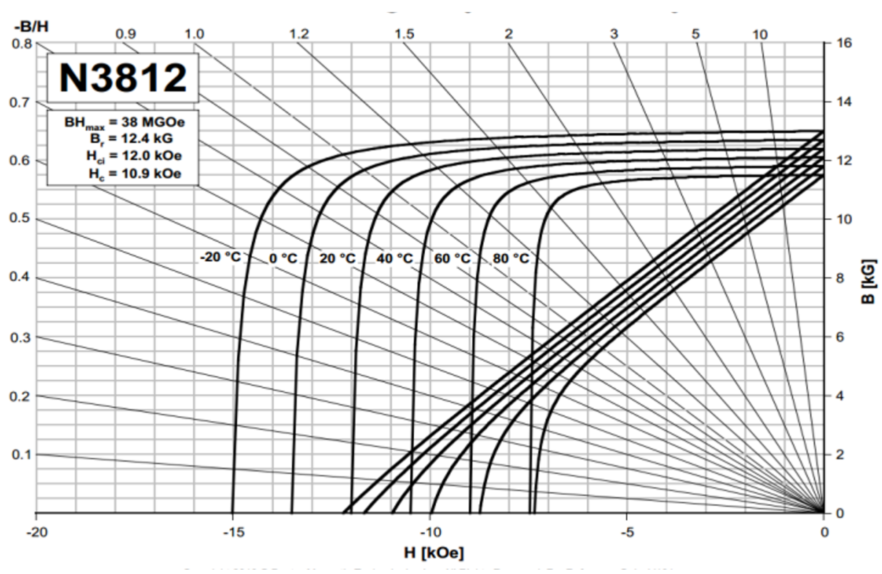


Figure 9.20: Magnet data sheet showing normal and intrinsic curves and varying temperatures.

Now suppose we increased the applied field, say to -11. We have now gone past the knee of the normal curve. Now when the external field is removed we follow the recoil path (parallel to the normal curve, but starting at maximum working point). The new working point (in the absence of an applied field) is now approx $H_{op} = -7$ with $B_{op} = 4$. We have therefore caused a permanent demagnetisation.

Suppose instead of applying an external field we increased the temperature of our magnet to 80°. In this case we can also see that the working point sits on the knee of the normal curve. At this temperature it would appear that any external field would push the working point over the knee and cause a permanent demagnetisation. Clearly the maximum admissible current depends on the temperature!

This is why it is important to design magnet systems in such a way that they can easily dissipate heat.

Part II

Cross-over Design

10 Introduction

We are interested in loudspeakers with broadband (preferably flat) frequency responses. So far we have been focusing on extending the low frequency response by modifying the enclosure (e.g. sealed, vented, transmission line, etc.)

It is important to acknowledge, however, that any particular loudspeaker has only a finite useful bandwidth. At low frequencies loudspeakers are limited for a number of reasons. Firstly, the radiation load. The real part of the radiation load dictates the radiated sound pressure (i.e. the energy dissipated). It is proportional to the square of wave number and driver radius. Consequently, we get greater radiation at high frequencies. I.e. it is harder to push air around at low frequencies. To compensate for this frequency dependence we have to make the surface area of the driver much greater (this is why low frequency loudspeakers have large drivers). At low frequencies the driver is also limited by its available travel. Some smaller drivers make up for their size by allowing a greater driver throw (i.e. larger amplitude of motion).

At high frequencies the motion of the driver is limited by the inertia of the moving mass (operating in the mass controlled region). Large loudspeakers are heavy; moving a heavy mass very quickly (i.e. at HF) is difficult! This limits the output power of the loudspeaker.

At high frequencies the inductance of the voice coil also comes into play, increasing the electrical impedance of the driver. This has the effect of acting like a low pass filter.

Before we even reach these high frequency issues loudspeakers begin to exhibit non-ideal behavior. Firstly, large drviers are very very directional at high frequencies, making them unsuitable. This is why smaller drivers, i.e. tweeters, are typically used for high frequency radiation. The low mass of the tweeter also avoids the issue of extra mass! The other problem is that at high frequencies loudspeakers no longer behave as rigid pistons. At high frequencies the driver starts to exhibit flexible modes (i.e. cone break up). This causes a very peaky response with respect to frequency. This is something we want to avoid.

So what is the solution? Well we have accepted the fact that different sized speakers are better at different frequencies. So why not use more than one loudspeaker? Why not split the frequency range into several smaller ranges, and used a different loudspeaker for each? This is the general idea behind mutli-driver loudspeakers. At low frequencies we want to use a large driver, which can displace lots of air, i.e. a woofer. In the mid frequency range we use 'squawkers' which is small than a woofer. Finally at high frequencies we use a tweeter. So as the frequency range increases, we use a small loudspeaker driver to radiate it. This reduces the moving mass, and also improves the overall directivity.

Although this sounds like a lot of effort, it is much easier than trying to design

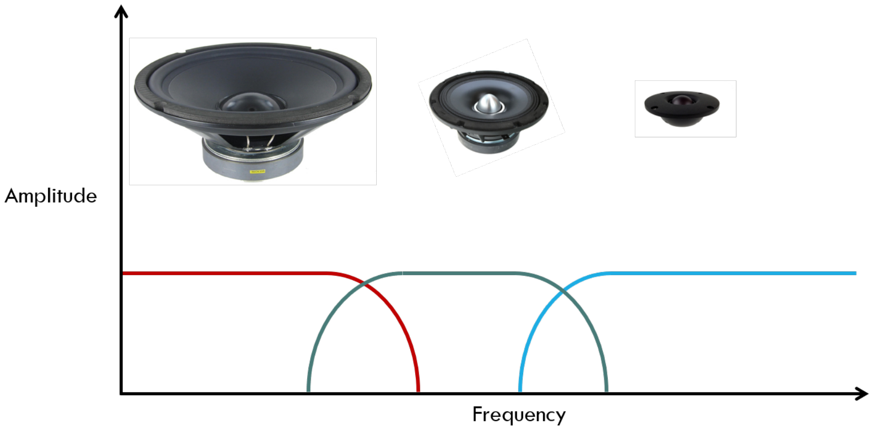


Figure 10.1: Cross-over idea - divide and conquer.

a single driver that covers the full range 20Hz to 20kHz.

A loudspeaker design engineer will do his/her best to design a loudspeaker that has a flat frequency response over a certain frequency range. To design a multi-driver system, we want the response of a driver to fall off in a smooth controlled way (we might want this roll off to be quite quick!). The point is that as one loudspeaker begins to roll off, another loudspeaker will begin to 'roll up'. These two contributions will cross-over at some frequency (the crossover frequency). Hopefully, if everything is designed correctly, the two drivers will support each other over this region and lead to a flat frequency response.

So how do we design this sort of cross over? It might be possible to use each drivers electro-acoustic design, changing the mass, damping and stiffness to achieve a particular roll off.. But this will make for a rather in flexible product.

Instead, it is easier to use electronic filters to control what happens over this cross-over region. E.g. we choose a driver that works well at low frequencies, and apply an electronic low pass filter to its input signals. We can then design a complementary filter that slow introduces a second mid ranger driver to take over at the cross over frequency.

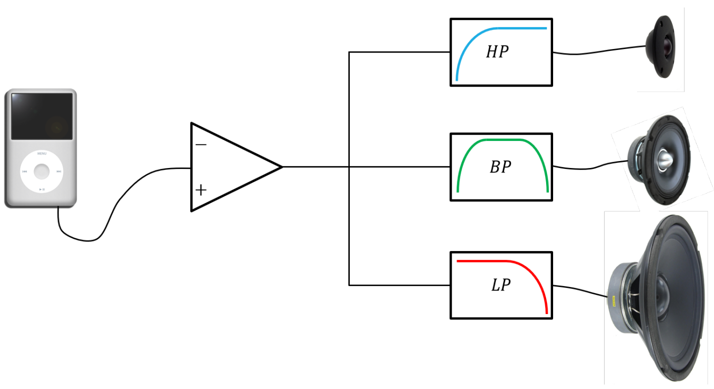


Figure 10.2: Passive crossover design.

So in general there are two ways of designing cross-overs, active or passive:

- Passive crossovers use only passive components (i.e. resistors/capacitors/inductors)

- Active crossovers use op-amps/other types of amplifiers to carry out the filtering.

Another important difference between active and passive crossovers is where you place the amplifier, before or after the crossover.

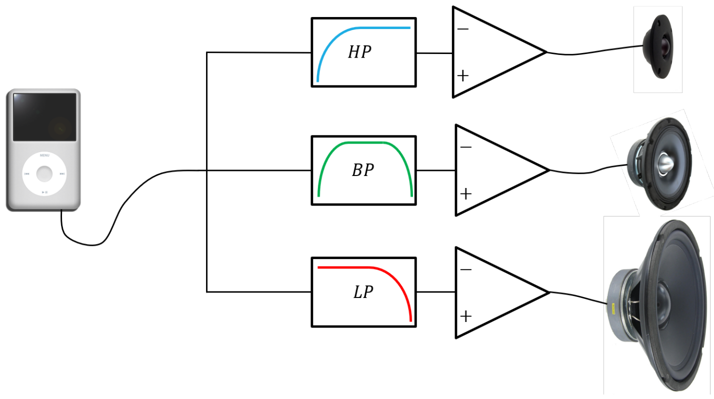


Figure 10.3: Active crossover design.

So what are the advantages/disadvantages?

The filters of a passive crossover must be capable of working with large signal amplitudes. This can cause some problems with regards to heat dissipation (there are always some losses). This requires some good plumbing in high power applications. This is how it is usually done in standard domestic Hi-Fi and TV/Radio. In an active crossover, the filters are not subject to large amplitudes and so power dissipation via heat is less of a problem. This is the main advantage of active type crossovers.

So what's the disadvantage of active crossovers? Well, we need to have an amplified for each driver! This can lead to a very large increase in the cost of a unit. This is why active crossovers are only seen in high end Hi-Fi or in high power PA systems.

Most Hi-Fi systems use passive crossovers, due to their cheep cost. The output of an amplifier is plugged straight into the back of the loudspeaker cabinet, inside which there is crossover network doing the high/low pass filtering.

Although passive filters are cheap and simple to design, active filters offer superior flexibility, you can design any frequency or phase response you like! There is also no need to worry about the impedance loading of the amplifiers (passive components may underload the amp). Active filters also have less losses, and can even be used to compensate for delays. This superior flexibility however, comes at a cost. Also, active filters require an external power supply.

There are lots of ways to design filters, but there are three main design choices; the type, cut-off and order. The type of filter describes the general shape of the filters frequency response, i.e. low pass, high pass, band pass, band stop, etc. The cut off frequency describes the frequency at which the filter begins to act on the signal (e.g. the minus 3dB point of a low pass filter). This frequency is tuned by adjusting the circuit's component values. The order of a filter determines the steepness of the roll-off, as well as the complexity of the circuit.

Designing a crossover filter is all about compromise. Too low an order and the roll off is too shallow and there will be significant overlap of the drivers operation. This can cause phase issues due to the different positions of the drivers. If the order is too high you can get unwanted artefacts in the phase, timing and transient response of the system. E.g. a high order/a steep roll off in frequency will cause a ringing effect at the resonance of the system. Also, high order filters require a greater number of components.

11 Practical Filter Design and the Prototype Filter

At the heart of any cross-over is a pair (perhaps more in the case of a multi-driver loudspeaker) of matched high/low pass filters, design such that their combined output provides a smooth transition between the two driver's operating regimes. Towards the beginning these notes we looked at some simple high/low pass filter circuit designs (see figures 3.9 and 3.11). Before reading on from here I would recommend going back over these to refresh your filter memory!

Now assuming you memory have been refreshed, lets start off by looking at what happens when you connect two drivers in parallel to the same amplifier. In practice amplifiers are designed to have a very very small source impedance. This means that the voltage does not drop when a load is applied. This means that if we connect multiple filters in parallel (i.e. changing the amplifier load), the voltage supplied doesn't change (the filters are uncoupled). This is good, we want the amplifier to behave as a constant voltage source. Lets look at an example.

In a two way passive loudspeaker system, the input to each loudspeaker will be connected to the output of an appropriate cross-over filter (high pass or low pass). The input of each cross-over filter will be connected to the output of a single amplifier. Together, the loudspeaker drivers, cross-over filters, and amplifier form single circuit, as in figure 11.1, where the amplifier is represented by

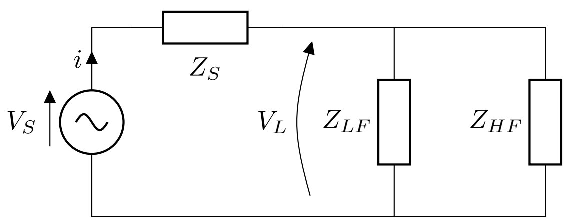


Figure 11.1: Low and high frequency drivers (including cross-over filters) connected in parallel to an amplifier with internal impedance Z_S.

an ideal voltage source V_S with a series impedance Z_S, and the low/high frequency drivers, along with their associated cross-over filters, are represented by the parallel impedances Z_{LF} and Z_{HF}.

The total load on the amplifier is simply the combined driver impedance,

$$Z_L = \frac{Z_{LF} Z_{HF}}{Z_{LF} + Z_{HF}}. \tag{11.1}$$

The voltage supplied to the parallel drivers V_L can now be determined using the potential divider rule,

$$V_L = V_S \frac{Z_L}{Z_S + Z_L}. \tag{11.2}$$

Now we can see that if the source impedance is very very small $Z_S \approx 0$, the above fraction is approx. 1. In this case the voltage across the loudspeaker V_L is the same as that of the ideal voltage source V_S. There is no effect on the output voltage as more speakers are connected in parallel.

This makes the design of cross-overs much easier, as we can design each filter/driver circuit separately. I.e. we don't have to worry about their interaction.

11.1 Second Order Low Pass

So far we have only considered simple first order low/high pass filters. Although nice and simple, first order filters are not typically used in cross-over applications. Their attenuation is too gradual, they don't cross-over quick enough! We are interested in filters with more rapid attention, i.e. high order. Lets start by look at a more useful second order filter, starting with the low pass.

Shown in figure 11.2 is a second order low pass filter, with a loudspeaker driver of impedance Z_{LS} attached to its output. Previously when we looked filters, we didn't actually consider what they would be connected to, i.e. we didn't include the effect of the loudspeaker driver! Now lets derive the transfer function V_{out}/V_{in}

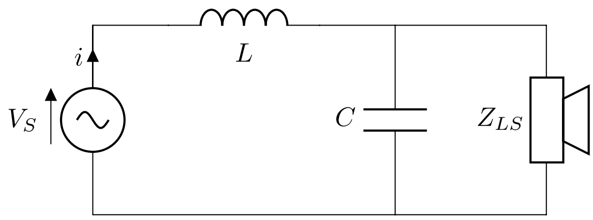


Figure 11.2: Second order low pass filter with loudspeaker loading

of this filter. The total output impedance Z_{out} will be the parallel combination of the capacitor and loudspeaker impedance,

$$Z_{out} = \frac{Z_C Z_{LS}}{Z_C + Z_{LS}}. \tag{11.3}$$

Recall the potential divider rule, and substitute in the input/output impedance,

$$\frac{V_{out}}{V_{in}} = \frac{Z_{out}}{Z_{in} + Z_{out}} = \frac{\frac{Z_C Z_{LS}}{Z_C + Z_{LS}}}{Z_L + \frac{Z_C Z_{LS}}{Z_C + Z_{LS}}}. \tag{11.4}$$

Now we want to rearrange this equation to get a 1 in the numerator. We can do this by multiplying the top and bottom by one over the numerator,

$$\frac{V_{out}}{V_{in}} = \frac{\frac{Z_C Z_{LS}}{Z_C + Z_{LS}}}{Z_L + \frac{Z_C Z_{LS}}{Z_C + Z_{LS}}} \times \frac{\frac{Z_C + Z_{LS}}{Z_C Z_{LS}}}{\frac{Z_C + Z_{LS}}{Z_C Z_{LS}}} = \frac{1}{\left(Z_L + \frac{Z_C Z_{LS}}{Z_C + Z_{LS}}\right) \frac{Z_C + Z_{LS}}{Z_C Z_{LS}}}. \tag{11.5}$$

Now its just a case of simplifying this equation. Ill leave it to you to go through the steps. You should end up with,

$$\frac{V_{out}}{V_{in}} = \frac{1}{\frac{Z_L}{Z_{LS}} + \frac{Z_L}{Z_C} + 1}. \tag{11.6}$$

Now lets substitute in the impedance for our inductor and capacitor,

$$\frac{V_{out}}{V_{in}} = \frac{1}{\frac{j\omega L}{Z_{LS}} + (j\omega)^2 LC + 1}. \tag{11.7}$$

Voila! We have the transfer function of our filter. Notice anything interesting? It depends on the loudspeaker impedance! This is very important. The loudspeaker acts as part of the filter circuit.

So, the filter transfer function depends on the loudspeaker impedance. Remember, the loudspeaker impedance depends on the driver, loading and the cabinet. It is basically some complex frequency dependent function. So does this mean that we need to design new crossover for every device? Usually what we do is replace the complex loudspeaker impedance Z_{LS} with its nominal electrical resistance R_{LS}. This is the value quoted by manufacturers. It is usually measured as the lowest point above the free air resonance in the loudspeaker's electrical impedance. This everything a lot simpler as loudspeakers are typically designed with standard nominal impedances (e.g. 4, 8, 16 ohms).

Later we will look at what effect simply ignoring the complex frequency dependent part of the loudspeaker impedance has on the performance of the cross over, and how we might attempt to correct for this. For now we will proceed on the assumption that our loudspeaker driver behaves like a nice simple resistor.

11.2 Cross-over Alignment

Now that we have derived the transfer function of our cross-over filter the next job is to determine the necessary component values to give us the response shape we want. This involves also determining the cross-over frequency of our two filters.

If we want to use a pair of low/high pass filters two smoothly transition between two drivers, we need to be able to find their cut off frequencies, so that we can align them appropriately.

Lets derive the -3dB point of the low pass filter we just derived. Now remember, -3dB is equivalent to the half power point. So we are interested in finding the frequency ω_c where the power (proportional to the magnitude squared transfer function) is half,

$$\left|\frac{V_{out}}{V_{in}}\right|^2 = \left|\frac{1}{\frac{j\omega_c L}{R_{LS}} + (j\omega_c)^2 LC + 1}\right|^2 = \frac{1}{2}. \tag{11.8}$$

Recall that $|Z|^2 = \Re(Z)^2 + \Im(Z)^2$, so

$$\frac{1}{2} = \frac{1}{(1-\omega_c^2 LC)^2 + \left(\frac{\omega_c L}{R_{LS}}\right)^2} = \frac{1}{1 - 2\omega_c^2 LC + \omega_c^4 L^2 C^2 + \frac{\omega_c^2 L^2}{Z_{LS}^2}} \tag{11.9}$$

which after group terms in ω_c gives us,

$$\frac{1}{2} = \frac{1}{1 + \omega_c^2\left(\frac{L^2}{R_{LS}^2} - 2LC\right)\omega_c^4 L^2 C^2}. \tag{11.10}$$

So we now derived an expression that is only satisfied when the filter response is -3dB. We want to determine the capacitor and inductor values that give us a particular -3dB cut of frequency ω_c. At the moment we have 2 unknowns (L and C) but only one equation. This isn't enough! We need another equation.

The issue is that there are lots of different types of filter that satisfy this equation. To determine our capacitor and inductor values we need to decide on the type of filter we want! There are a whole load of different filter types out there. Luckily, we are only interested in one particular type: the Butterworth.

The Butterworth filter is particularly popular because its response shape (also called an alignment) has the unique property that is it *maximally flat* (i.e. extends as far as possible without introducing ripple). In general the Butterworth alignment satisfies the equation,

$$\left|\frac{V_{out}}{V_{in}}\right|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_c}\right)^{2n}} \tag{11.11}$$

where n is the order of the filter. Together, equation 11.10 and 11.11 give us two equations. We can use these to determine the capacitor and inductor values that not only give us a -3dB of frequency ω_c, but also satisfy the Butterworth alignment.

In order that our filter transfer function equation satisfies the 2nd order Butterworth alignment it is clear that we must have that,

$$\omega_c^2\left(\frac{L^2}{R_{LS}^2} - 2LC\right) = 0 \tag{11.12}$$

and

$$\omega_c^4 L^2 C^2 = 1. \tag{11.13}$$

This gives us two simultaneous equations, one of which can be solved to find the cut off frequency directly,

$$\omega_c^2 = \frac{1}{LC}. \tag{11.14}$$

We can also take these two equations and derive equations for the capacitor and inductor values that give a desired cut off frequency. Here are the steps:

Rearrange equation 11.13 to obtain the capacitance C,

$$\omega_c^4 L^2 C^2 = 1 \rightarrow C^2 = \frac{1}{\omega_c^4 L^2} \rightarrow C = \frac{1}{\omega_c^2 L}. \tag{11.15}$$

From equation 11.14 substitute $LC = 1/\omega_c$ into equation 11.12,

$$\omega_c^2\left(\frac{L^2}{R_{LS}^2} - 2\frac{1}{\omega_c^2}\right) = 0 \rightarrow \omega_c^2\frac{L^2}{R_{LS}^2} - 2 = 0 \tag{11.16}$$

and rearrange for the inductance,

$$\omega_c^2\frac{L^2}{R_{LS}^2} = 2 \rightarrow \frac{L^2}{R_{LS}^2} = \frac{2R_{LS}^2}{\omega_c^2} \rightarrow L = \frac{\sqrt{2}R_{LS}}{\omega_c}. \tag{11.17}$$

Now substitute $L = \frac{\sqrt{2}R_{LS}}{\omega_c}$ into $C = \frac{1}{\omega_c^2 L}$,

$$C = \frac{1}{\sqrt{2}R_{LS}\omega_c}. \tag{11.18}$$

Equations 11.17 and 11.18 give us the inductor and capacitor values required to achieve a Butterworth alignment with a -3dB point of ω_c, taking into acount the nominal DC resistance of the loudspeaker driver, R_{LS}.

Shown in figure 11.3 is an example response of a Butterworth filter, obtained by substituting equations 11.17 and 11.18 into equation 11.7,

$$\frac{V_{out}}{V_{in}} = \frac{1}{\frac{j\omega L}{Z_{LS}} + (j\omega)^2 LC + 1} \rightarrow \frac{1}{1+\sqrt{2}\left(\frac{j\omega}{\omega_c}\right)+\left(\frac{j\omega}{\omega_c}\right)^2}. \tag{11.19}$$

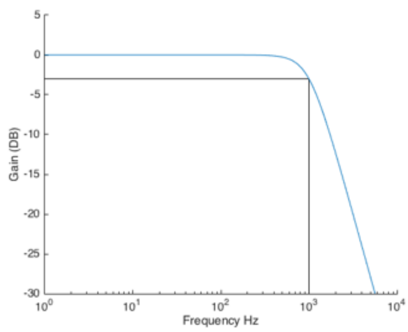


Figure 11.3: Magnitude response of a 2nd order low pass Butterworth filter (cut-off frequency is marked in black).

Equation 11.19 represents a parametrised from of the Butterworth filter transfer function. If you were to take its magnitude and square it, you would end up with exactly equation 11.11. Have a go!

Although the above analysis wasn't particularly hard... it wasn't exactly fun. Also, for more complicated filters things get much much more horrible. We had two simultaneous equations because we had two component values... Higher order filter means more components, more components means more equations..! So is there another way of designing filters? Yes! We can use tables of normalised filter coefficients! But you will have to wait until later for this. Right now we want to revisit our assumption that $Z_{LS} = R_{LS}$.

11.3 Complex Loudspeaker Impedance

So we made life a little easier for ourselves by assuming the loudspeaker impedance was just the nominal resistance. We know in reality the true response is much more complex.

So we have made life a little easier for ourselves by assuming the loudspeaker impedance is just the nominal resistance. We know in reality the true response is much more complex.

Shown in figure 11.4 is a comparison of the gain of a second order low pass filter with a nominal resistive load and with a frequency dependent loudspeaker impedance, including all of the electrical, mechanical and acoustical loads. Turns out what we get is quite good. We can see that there isn't a huge difference between the two.

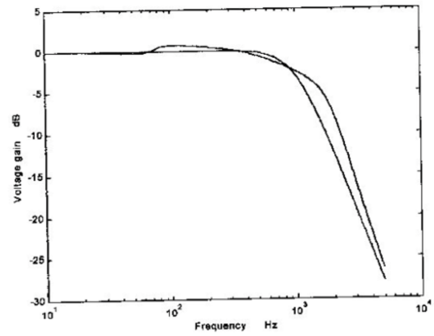


Figure 11.4: Low pass filter response with nominal resistive load, and complex loudspeaker load.

By assuming a nominal impedance we get very close to the correct answer. The true response is a little bumpier and the roll off is more complex, and there are some marginal differences in the –3dB point.

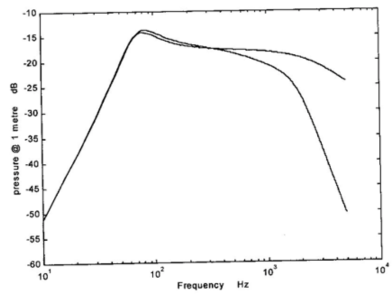


Figure 11.5: Radiated acoustic pressure with and without low pass filter.

Perhaps it is more interesting to understand what happens at the loudspeaker output? If we include a pistonic source model we can compare the acoustic response with and without the cross-over. We can see that the cross-over filter rolls of the response quite smoothly (the cut off is around 1 kHz). While there is some difference near to the driver resonance, it really isn't very large. So even by assuming this nominal speaker impedance, we get an answer that isn't too terrible!

A little later on we will look at how we can improve our results even more, by compensating for the loudspeakers complex impedance.

11.4 Low Pass Prototype

Okay, we have been through the analysis of a 2nd order low pass filter. But what about other filter types? Different orders? High pass?

Filters are usually designed based on what is called a low pass prototype, and it turns out that we have already been through the analysis of this filter (figure 11.2).

To convert this low pass prototype into a high pass, all we have to do is interchange the capacitors and inductors, as in figure 11.6. As an exercise have a go at deriving the transfer function for this filter, including the loudspeaker loading. You should get,

$$\frac{V_{out}}{V_{in}} = \frac{1}{\frac{Z_C}{Z_{LS}} + \frac{Z_C}{Z_L} + 1} = \frac{1}{\frac{1}{j\omega C Z_{LS}} + \frac{1}{(j\omega)^2 CL} + 1}. \tag{11.20}$$

Now providing we still want Butterworth alignment, the equations we derived

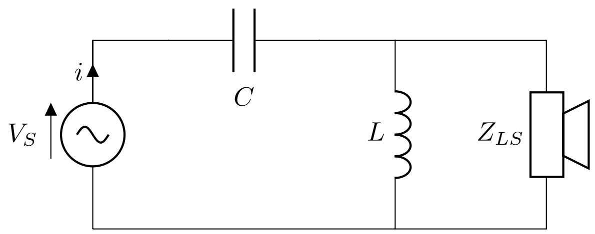


Figure 11.6: Second order high pass filter with loudspeaker loading.

for the capacitance and inductance (11.18 and 11.17) are still valid (have a go re-deriving them from the equation 11.20).

Substituting the L and C values for the Butterworth alignment into equation 11.20 yields,

$$\frac{V_{out}}{V_{in}} = \frac{1}{\frac{1}{j\omega C Z_{LS}} + \frac{1}{(j\omega)^2 CL} + 1} \rightarrow \frac{1}{1 + \sqrt{2}\left(\frac{\omega_c}{j\omega}\right) + \left(\frac{\omega_c}{\omega}\right)^2}. \tag{11.21}$$

Notice that compared to equation 11.19, all that has changed is that where we previously had $(j\omega/\omega_c)$, we now have $(\omega_c/j\omega)$. This has the effect of reversing the frequency response!

So to design a high pass filter, we begin by designing a prototype low pass filter, then we interchange the inductor and capacitor. We can also use the low pass prototype to design higher order filters.

11.5 Higher Order Filters

A second order filter will have a roll off slope of -12dB/oct. This is because of the ω^2 term in its transfer function. Often we will want to roll off the contribution of each loudspeaker more rapidly. To do so we need to increase the order of the filter used.

The order of a filter's transfer function is directly related to the number of *reactive* components in the filter. We can see this by considering our second order high pass (figure 11.6).

The transfer function of our 2nd order high pass is given by,

$$\frac{V_{out}}{V_{in}} = \frac{1}{\frac{1}{j\omega C Z_{LS}} + \frac{1}{(j\omega)^2 CL} + 1}. \tag{11.22}$$

Below the cut off frequency ($\omega_c = 1/LC$) the magnitude of this transfer function increase at a rate of 12dB/oct, due to the $1/\omega^2$ term in its denominator. Now suppose we increase the impedance of the parallel inductor such that it tends to infinity $L \rightarrow \infty$, i.e. no current can pass through this element. This is equivalent to removing the inductor from the circuit all together. As the inductance L tends toward infinity the term $1/(j\omega)^2 CL$ tends towards zero. Now the only frequency dependence left is that of the $1/\omega$. This corresponds to a 6dB/oct rise below the cut off frequency. Hence the roll of rate is related to the number of reactive elements in the circuit.

There are several different ways or creating higher order circuits by introducing additional reactive components. A particularly simple approach is to use a ladder structure (or a Cauer topology). The idea it to chain together several low order filters, passing the output of one, into the input of another. Lets look at an example using our low pass prototype.

Shown in figure 11.7 is a 6th order low pass filter. Three second order low pass prototype filters have been cascaded, with the input of one set to the output of the previous. Each additional low pass prototype introduces two additional reactive components, and so increases the filter order by 2.

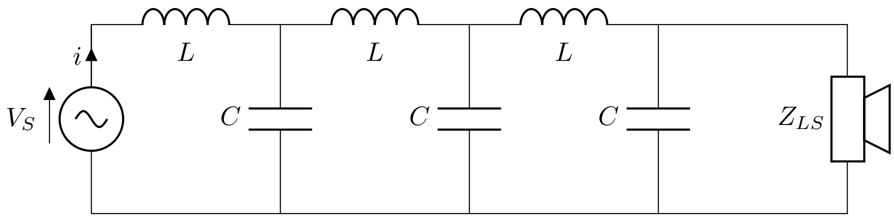


Figure 11.7: Second order high pass filter with loudspeaker loading.

Now lets work through the analysis of this circuit and derive the equations for each capacitor and inductor...

Only joking! That would be horrible. We don't want to have to analyse by hand large filters like that. Instead we can use filter design tables! This is way more practical.

Shown in figure 11.8 are a pair of stacked low pass filters. They differ in that the top circuit begins with a shunt capacitor, whilst the bottom circuit begins with a series inductor. They are however, both low pass filters. We are interested in determining the capacitor and inductor values necessary to obtain a Butterworth filter alignment with a cut off frequency of ω_c. Rather than deriving by hand all the equations and solving these by hand, we can use tables of *normalised* filter coefficients, as in figure 11.1. Here, for a given order of filter (1 to 10 are shown) the table values dictate the component values, depending on whether the shunt capacitor or series inductor circuit is used.

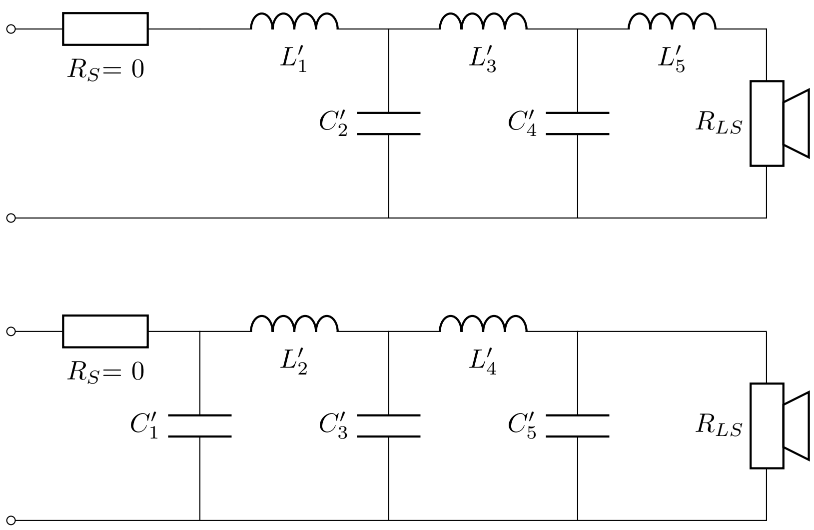


Figure 11.8: Stacked low pass filter topologies for normalised coefficients in table 11.1.

Lets look at an example. We have already gone through the analysis of the 2nd order low pass filter starting with a series inductor. We derived the values of L and C as,

$$L = \frac{\sqrt{2}R_{LS}}{\omega_c} \tag{11.23}$$

$$C = \frac{1}{\sqrt{2}R_{LS}\omega_c}. \tag{11.24}$$

Notice that the factors of $\sqrt{2} = 1.414$ and $1/\sqrt{2} = 0.707$ happen to coincide with the normalised filter coefficients in figure 11.1!

From inspection we can see that these coefficients are normalised in two ways:

1) so that the impedance that the filter expects to see (i.e. of the loudspeaker) is 1 ohm

2) and so that the cut off of the LP filter is 1 rad/s.

To calculate the necessary component values we need to denormalise these coefficients (i.e. impedance scaling and frequency denormalisation). The frequency denormalisation scales the components values to move the cut off frequency up or down, whilst the impedance scaling makes sure that the circuit expects the correct loading.

For the impedance scaling we just scale the impedance of each component by R_{LS} (e.g. 8 ohm). Remeber, the impedance of a capacitor is $1/j\omega C$, so we divide the capacitance by R_{LS}. For the frequency denormalisation we divide the reactive components by new cut off frequency. Based on the above, the components values are given by,

$$C = C_n \times \frac{1}{\omega_c R_{LS}} \tag{11.25}$$

and

$$L = L_n \times \frac{R_{LS}}{\omega_c} \tag{11.26}$$

where C_n and L_n are the normalised coefficients from figure 11.1 (or equivalent). These values will give us the low pass filter we want! It is important to note that the coefficient given in figure 11.1 are for a Butterworth alignment. Other tables are available in filter design books for more complex filter types. Also, the values given assume that the source impedance is 0, $R_S = 0$. This should be approximately true for a well designed amplifier.

Order	C_1	L_2	C_3	L_4	C_5	L_6	C_7	L_8	C_9	L_{10}
1	1.00000									
2	1.41422	0.70711								
3	1.50000	1.33333	0.50000							
4	1.53074	1.57716	1.08239	0.38268						
5	1.54509	1.69443	1.38196	0.89443	0.30902					
6	1.55292	1.75931	1.55291	1.20163	0.75787	0.25882				
7	1.55765	1.79883	1.65883	1.39717	1.05496	0.65597	0.22521			
8	1.56073	1.82464	1.72874	1.52832	1.25882	0.93705	0.57755	0.19509		
9	1.56284	1.84241	1.77719	1.62019	1.40373	1.14076	0.84136	0.51555	0.17365	
10	1.56435	1.85516	1.81211	1.68689	1.51000	1.29209	1.04062	0.76263	0.46538	0.15643
$R_S = 0$	L'_1	C'_2	L'_3	C'_4	L'_5	C'_6	L'_7	C'_8	L'_9	C'_{10}

Table 11.1: Table of normalised filter coefficients for a Butterworth alignment.

So now, with the help of figure 11.1, our streamlined process for designing a low pass filter is as follows:

1) Decide on circuit topology (shunt capacitor or series inductor)

2) Decide on filter order

3) Choose filter alignment (and find appropriate coefficient table)

4) Get normalised capacitor and inductor coefficients

5) Denormalise and rescale coefficients to get component values

Now what about high pass filters? Well, we have already seen that by interchanging the capacitor and inductor in a low pass filter, we obtain an equivalent high pass filter with the same cut-off frequency. So this is exactly what we do. We design a low pass filter with the cut off frequency we want, find the component values, then interchange their positions. Job done.

So far we have been considering the low pass and high pass filters of our cross-over independently. Whilst this is perfectly okay to do (providing the amplifier impedance is sufficiently small!), we mustn't forget what we are trying to achieve with our cross-over. We are trying to design a pair of filters which allow us to smoothly transition between two drivers. To achieve a smooth transition between any pair of drivers the combined output of our cross-over must provide a uniform response, even in the absence of any drivers. I.e. we need to check that when the outputs of our two filters (low pass and high pass) are combined, they yield a uniform response. This will be the focus of the next chapter.

12 Combining Filter Outputs

To achieve a smooth transition between a pair of drivers it is necessary that, when combined, the outputs of our cross-over provide a uniform response.

At sufficiently low and high frequencies, i.e. when the two filters are well into in their pass and stop bands, the signal sent to one driver will be heavily attenuated, whilst the other is unattenuated. The total acoustic response is then just that of the unattenuated driver.

Things are less obvious in the vicinity of the cross-over frequency (i.e. the cut-off frequency of the two filters). In this region the two signals are of similar amplitude, and so both drivers contribute to the total acoustic response.

To achieve a flat frequency response over this cross-over region we need the low and high pass filters to combine appropriately. Whilst in reality it is the combined acoustic response of the two drivers that we want to be flat, we can safely assume that these drivers are operating in their reference regions, and so their input signals are transduced 'perfectly'.

The combined voltage output of the cross-over is given by,

$$v_{tot} = H_{lp}v_{in} + H_{hp}v_{in} = (H_{lp} + H_{hp})\, v_{in} = H_{tot}v_{in} \tag{12.1}$$

where H_{lp} and H_{hp} are, respectively, the low and high pass filter transfer functions, and the combined transfer function H_{tot} is given by,

$$H_{tot} = H_{lp} + H_{hp}. \tag{12.2}$$

In what follows we will look at what happens when we combine the outputs of different order Butterworth filters.

12.1 Butterworth Transfer Functions

Recall that the Butterworth alignment is characterised by the magnitude squared transfer function,

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}} \tag{12.3}$$

where n is the filter order and ω_c is the cut-off (-3dB) frequency. It is easy to see that at the cut off frequency, $\omega = \omega_c$, i.e. when $\omega/\omega_c = 1$ the power is equal to $1/2$ (i.e. the magnitude is -3dB or 0.707). This is true of any order! Shown in figure 12.1 are the Butterworth magnitudes for various orders. Figure 12.1 looks quite promising for a cross-over filter. The high pass version will be identical just flipped horizontally around cut-off frequency. It looks like if we add the output of the low pass and high pass we might get something that looks pretty flat? This is what we want! So fingers crossed...

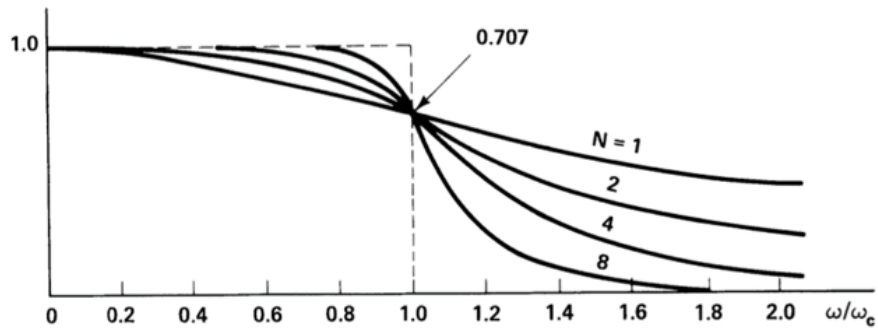


Figure 12.1: Butterworth transfer functions of increasing order.

What are we missing though? Phase! Figure 12.1 only shows the transfer function magnitude. Phase is essential when looking at the summation of any complex responses. So how do we get the complex transfer function H of the Butterworth filter from its magnitude squared transfer function $|H|^2$? That requires a trip to the Laplace domain. To avoid this interesting, but long winded expedition I will just give you the complex transfer functions.

Here are the transfer functions for the first three orders of low and high pass Butterworth filters. For higher order filters the denominator polynomials can get a little complicated! You can find general expressions for them online if you are curious. Notice that in converting from low pass to high pass all we have to do

Order	Low Pass	High Pass
1	$\frac{1}{1+\left(\frac{j\omega}{\omega_c}\right)}$	$\frac{1}{1+\left(\frac{\omega_c}{j\omega}\right)}$
2	$\frac{1}{1+\sqrt{2}\left(\frac{j\omega}{\omega_c}\right)+\left(\frac{j\omega}{\omega_c}\right)^2}$	$\frac{1}{1+\sqrt{2}\left(\frac{\omega_c}{j\omega_c}\right)+\left(\frac{\omega_c}{j\omega}\right)^2}$
3	$\frac{1}{\left(1+\left(\frac{j\omega}{\omega_c}\right)\right)\left(1+\left(\frac{j\omega}{\omega_c}\right)+\left(\frac{j\omega}{\omega_c}\right)^2\right)}$	$\frac{1}{\left(1+\left(\frac{\omega_c}{j\omega}\right)\right)\left(\left(\frac{\omega_c}{j\omega}\right)+\left(\frac{\omega_c}{j\omega}\right)^2\right)}$

Table 12.1: Complex transfer functions of low and high pass Butterworth filters of order 1, 2 and 3.

is replace $j\omega/\omega_c$ with $\omega_c/j\omega$.

If you take the magnitude square of any of these transfer functions you will end up with the Butterworth equation in the top right. Going the other way, from the magnitude squared function to the transfer function is quite involved, and luckily, we don't need to do it!

Now that we have the complex transfer function for orders 1, 2 and 3, lets look at what happens when we add the *complex* outputs of two Butterworth filters, starting with the first order...

12.2 First Order Butterworth Summation

The easiest way to look at the summed response of two filters is to consider them as vectors in the complex plane. So lets draw them as such. We have our complex transfer functions,

$$H_{lp} = \frac{1}{1+\left(\frac{j\omega}{\omega_c}\right)} \tag{12.4}$$

and

$$H_{hp} = \frac{1}{1-j\left(\frac{\omega_c}{\omega}\right)}. \tag{12.5}$$

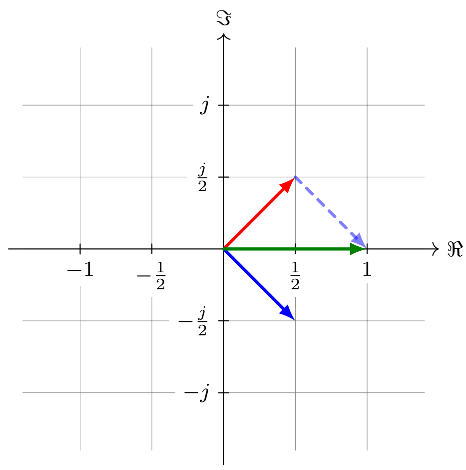


Figure 12.2: Low pass (blue), high pass (red), and combined (green) Butterworth responses in complex plane.

To plot these in the complex plane we need to rearrange them to separate real and imaginary parts. Lets also evaluate them at the cross-over frequency $\omega = \omega_c$.

$$\frac{1}{1+\left(\frac{j\omega}{\omega_c}\right)} \xrightarrow{\omega=\omega_c} \frac{1}{1+j} = \frac{1}{1+j}\left(\frac{1-j}{1-j}\right) = \frac{1}{2} - \frac{j}{2} \tag{12.6}$$

$$\frac{1}{1-j\left(\frac{\omega_c}{\omega}\right)} \xrightarrow{\omega=\omega_c} \frac{1}{1-j} = \frac{1}{1-j}\left(\frac{1+j}{1+j}\right) = \frac{1}{2} + \frac{j}{2} \tag{12.7}$$

The total combined response is then given by,

$$H_{tot} = \left(\frac{1}{2} - \frac{j}{2}\right)\left(\frac{1}{2} + \frac{j}{2}\right) = 1. \tag{12.8}$$

Visually we can interpret the above as the summation of two vectors in the complex plane, as in figure 12.2. Since the imaginary parts are equal and opposite, they cancel when added, and we are left with a real value of magnitude 1. Also note that the two filter responses are out of phase by $90°$.

So at the crossover frequency the two filters combine such their total gain is 1, or 0dB. What about rest of the frequencies? Shown in figures 12.3 and 12.4 are the filter's magnitude and phase responses across the entire frequency range. Also shown is the combined response (in black).

It is clear that the summed response is equal to 1 across the entire frequency range; we have a nice flat response! This is exactly what we would want from a cross-over filter.

Note however that the phase response of the two filters differ by $90°$ at all frequencies. This phase offset might cause us some problems... Remember that these filter responses correspond to those of a woofer and tweeter. These drivers will not be located in the same position, there will be some spacing between them. This means that there will be a path length difference between each of them and a receiver position. This will change with angle. At some angle, the phase shift caused by the path length difference will equal that of the filters phase offset. This will lead to constructive interference and the introduction of a peak in the amplitude! We will look at this a little more later on. For now, lets continue with out summed Butterworth responses.

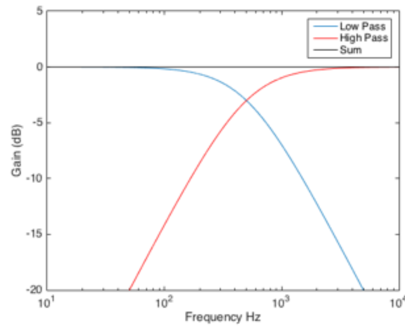


Figure 12.3: Low pass (blue), high pass (red), and combined (black) 1st order Butterworth magnitude response.

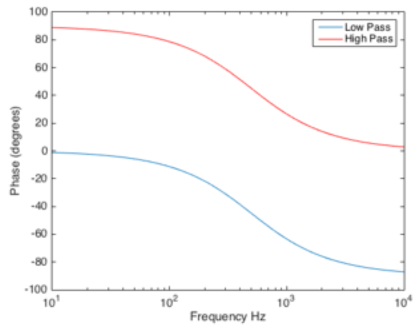


Figure 12.4: Low pass (blue) and high pass (red) 1st order Butterworth phase response.

12.3 Second Order Butterworth Summation

Now lets consider the summation of two second order Butterworth filters with the complex transfer functions,

$$H_{lp} = \frac{1}{1+\sqrt{2}\left(\frac{j\omega}{\omega_c}\right)+\left(\frac{j\omega}{\omega_c}\right)^2} \tag{12.9}$$

and

$$H_{hp} = \frac{1}{1+\sqrt{2}j\left(\frac{\omega_c}{\omega}\right)+\left(\frac{\omega_c}{j\omega}\right)^2}. \tag{12.10}$$

Evaluating the above at $\omega = \omega_c$ and separating the real and imaginary parts yields,

$$\frac{1}{1+\sqrt{2}\left(\frac{j\omega}{\omega_c}\right)+\left(\frac{j\omega}{\omega_c}\right)^2} \xrightarrow{\omega=\omega_c} \frac{1}{1+j\sqrt{2}-1} = -\frac{j}{\sqrt{2}} \tag{12.11}$$

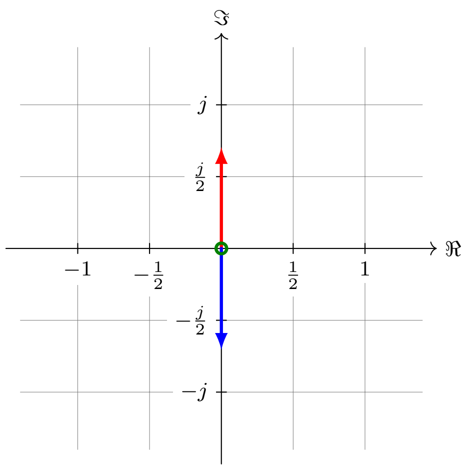
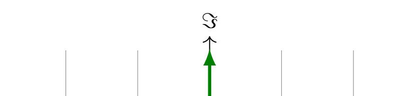


Figure 12.5: Low pass (blue), high pass (red), and combined (green) 2nd order Butterworth responses in complex plane.



and

$$\frac{1}{1+\sqrt{2}\left(\frac{j\omega}{\omega_c}\right)+\left(\frac{j\omega}{\omega_c}\right)^2} \xrightarrow{\omega=\omega_c} \frac{1}{1-j\sqrt{2}-1} = \frac{j}{\sqrt{2}}. \tag{12.12}$$

Notice that the transfer function outputs are purely imaginary, both with equal amplitude and opposite direction. I.e. they are out of phase. So what happens when we sum these responses? We get complete cancellation,

$$|H_{tot}| = \left|-\frac{j}{\sqrt{2}}+\frac{j}{\sqrt{2}}\right| = 0. \tag{12.13}$$

The response is 0! I.e. the gain is 0 or -∞ dB. This is shown visually in figure 12.5, where the filter outputs are again plotted as vectors in the complex plane.

So 2nd order Butterworths aren't looking like a very good choice for a crossover... But remember, the filters are used to drive loudspeakers and really it is the output of the loudspeaker that we are interested in. We could easily invert the wiring of one of the loudspeakers to flip the phase. This would give us two transfer function of equal magnitude AND direction, as in figure 12.6. I.e. they are in phase.

Now what is the total response? Well its,

$$|H_{tot}| = \left|\frac{j}{\sqrt{2}}+\frac{j}{\sqrt{2}}\right| = \frac{2}{\sqrt{2}} = 1.414. \tag{12.14}$$

So now we have a gain greater than 1. In dB we have $20\log_{10}(1.414) = 3$ dB. So at the cut off frequency, the combined response gets a 3dB boost! The full frequency response for the two cases above are shown in figure 12.7. On the left is the original filter summation (no phase inversion), and on the right we have the phase inverted response.

It is clear that for the original response the $180°$ phase difference leads to a massive cancellation at the crossover frequency. This is clearly not what we want from a crossover filter!

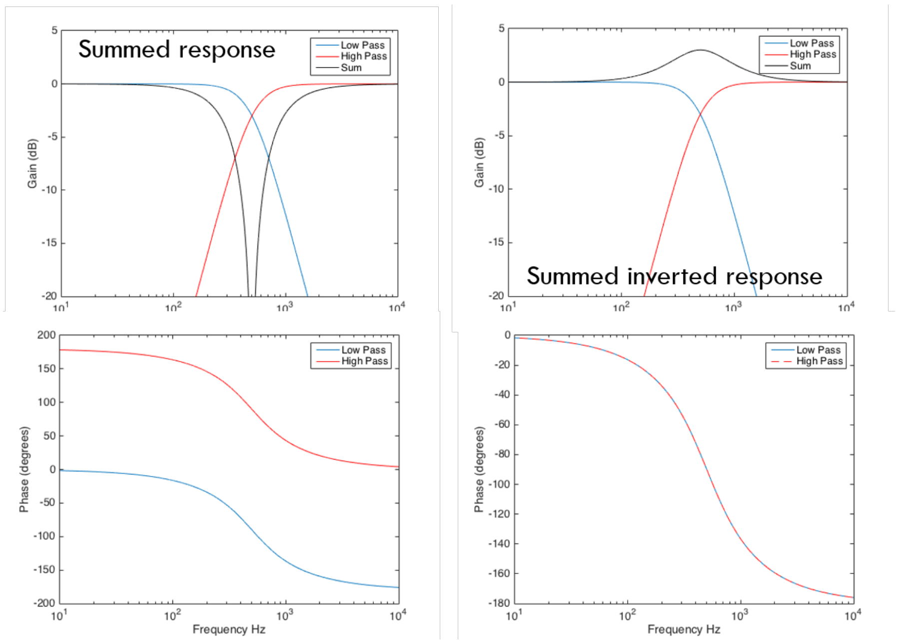


Figure 12.7: Low pass (blue), high pass (red), and combined (black) 2nd order Butterworth magnitude and phase responses.

On the other side we have the phase inverted response. Here the in-phase nature of the two responses mean that at the crossover frequency we get constructive interference, and 3dB boost. Again, this isn't really what we wanted. BUT the two filters are now in phase, which we like.

Notice that in either case, the roll off rate of the filters are much greater than the previous first order case, as expected.

Now lets cross our fingers and hope that the 3rd order filter gives us what we want (flat magnitude response and in phase!).

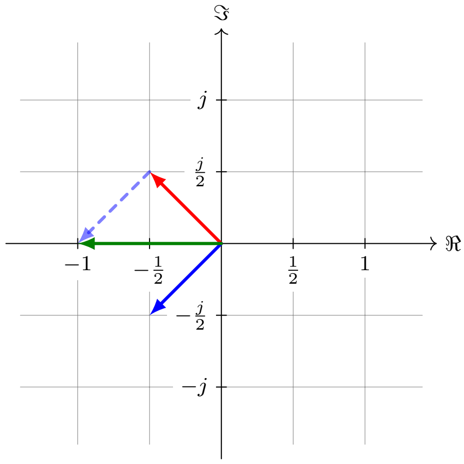


Figure 12.8: Low pass (blue), high pass (red), and combined (green) 3rd order Butterworth responses in complex plane.

12.4 Third Order Butterworth Summation

Now lets consider the summation of two third order Butterworth filters with the complex transfer functions,

$$H_{lp} = \frac{1}{\left(1+\left(\frac{j\omega}{\omega_c}\right)\right)\left(\left(\frac{j\omega}{\omega_c}\right)+\left(\frac{j\omega}{\omega_c}\right)^2\right)} \tag{12.15}$$

and

$$H_{hp} = \frac{1}{\left(1+\left(\frac{\omega_c}{j\omega}\right)\right)\left(\left(\frac{\omega_c}{j\omega}\right)+\left(\frac{\omega_c}{j\omega}\right)^2\right)}. \tag{12.16}$$

Evaluating the above at $\omega = \omega_c$ and seperting the real and imaginary parts yields,

$$\frac{1}{\left(1+\left(\frac{j\omega}{\omega_c}\right)\right)\left(\left(\frac{j\omega}{\omega_c}\right)+\left(\frac{j\omega}{\omega_c}\right)^2\right)} \xrightarrow{\omega=\omega_c} \frac{1}{(1+j)(1+j-1)} = \frac{1}{(1+j)j} = \frac{1}{j-1} = -\frac{1}{2}-\frac{j}{2} \tag{12.17}$$

and

$$\frac{1}{\left(1+\left(\frac{\omega_c}{j\omega}\right)\right)\left(\left(\frac{\omega_c}{j\omega}\right)+\left(\frac{\omega_c}{j\omega}\right)^2\right)} \xrightarrow{\omega=\omega_c} \frac{1}{(1-j)(1-j-1)} = \frac{1}{-(1-j)j} = \frac{1}{-(1+j)} = -\frac{1}{2}+\frac{j}{2}. \tag{12.18}$$

Notice that like the first order summation, the transfer function outputs have equal real parts, and opposite imaginary parts. Like the first order summation their sum yields a magnitude of 1,

$$|H_{tot}| = \left|\left(-\frac{1}{2}-\frac{j}{2}\right)+\left(-\frac{1}{2}+\frac{j}{2}\right)\right| = 1. \tag{12.19}$$

This can be seen visually in figure 12.8.

The full magnitude and phase response are shown in figures 12.9 and 12.10. As we might expect, the two filters combine excellently to give a constant magnitude response (notice the steeper roll off than before). But just like the first order summation, there is a constant $90°$ phase shift across the entire response. As before, this will lead to constructive (also destructive) interference at some particular angle where the additional path length due to the non-collocated drivers coincides with the filter phase offset.

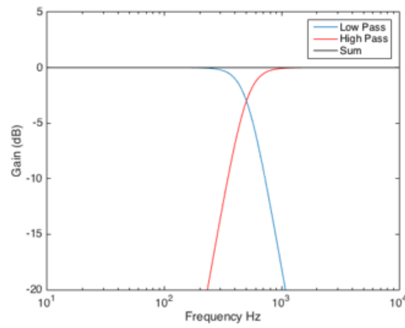


Figure 12.9: Low pass (blue), high pass (red), and combined (black) 3rd order Butterworth magnitude response.

So it appears we have come full circle. What do you expect would happen if we went up to fourth order? We would get something that looks a lot like a 2nd order summation! Basically, every time we increase the filter order, we introduce a factor of $j/2$ which has the effect of rotating the complex transfer function by $45°$ in the complex plane.

Now lets look a little more closely at this phase off-set issue.

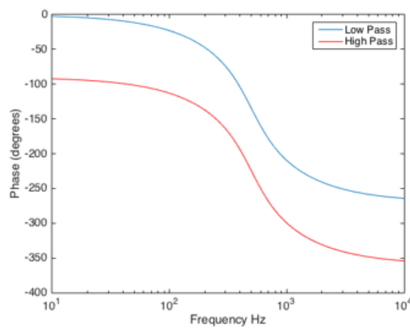


Figure 12.10: Low pass (blue) and high pass (red) 3rd order Butterworth phase response.

12.5 Implications - Butter'worth-less'?

Figure 12.11: Morpheus not caring about a 90° phase lag.

From the above we can conclude that:

1) Odd order filters give us a great magnitude response, but we get a 90° phase shift between the high pass and low pass.

2) Even order filters give us a great phase response, but we either get a huge cancellation or a 3dB boost at the cut off frequency.

So what's the issue with this 90° phase lag?

If we looked only at the magnitude response, we might be tempted to say that all we need is an odd ordered Butterworth filter and everything will be fine right? Well, the magnitude only tells us half the story. We have to consider the phase response also.

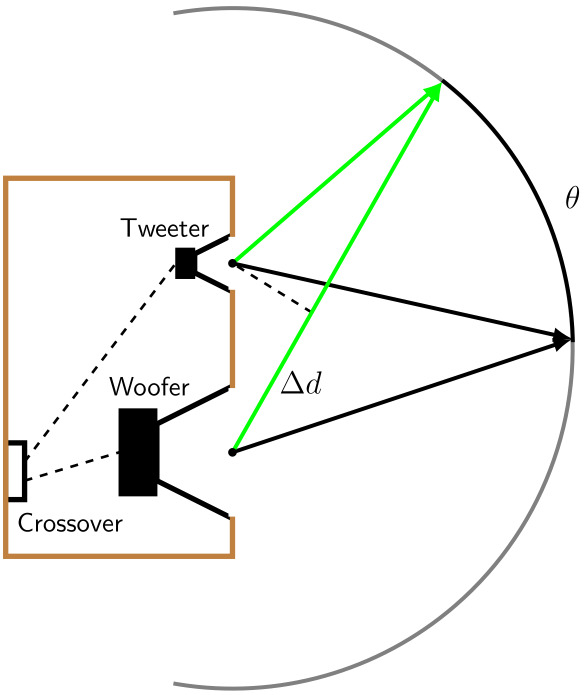


Figure 12.12: Path length difference between woofer and tweeter in a loudspeaker system.

In a typical loudspeaker system the woofer and tweeter are not coincident. This means that for any off-axis listener position their exists a path length difference between the two. This path length difference will vary with the vertical angle considered, and will introduce a phase different between the two driver contributions.

At some particular angle, the phase shift introduced by the path length difference will equal the 90° phase shift between the two filters (remember, odd order Butterworths have a 90° phase offset). Note however, that there will clearly be two of these angles! One above, and one below. In one, the woofer leads and the tweeter lags, and in the other the tweeter leads and the woofer lags. At these two angles the additional path length will either counteract or further shift the filters phase response, leading to either constructive and destructive interference.

What about the response on axis? Well on axis there is no additional phase shift, and so we get a gain of exactly 1. Lets look at an example.

If we take two 3rd order Butterworth filters with a cross-over frequency of 1.7 kHz and two drivers spaced apart by 17 cm we will see the following directivity in figure 12.13.

Here we have chosen to Ignores the directivity of each individual driver, and treat them as omnidirectional.

Because the two drivers are 90 degrees out of phase (due to the filter phase offset) on axis we have our desired 0dB response. At an angle of $15°$ however, the path length difference introduces an addition phase shift which causes the two drivers to become out of phase. This leads to destructive interference and a massive cancellation. This is the called the cancelation axis.

At $-15°$ we get the same path length difference, this time however the driver positions have been interchanged! As such the additional phase shift causes the two drivers to become in phase. This leads to constructive interference and a 3dB boost. This is called the oeaking axis.

The magnitude frequency response of the two drivers together are shown in figure 12.14 for the on-axis, cancellation axis, and peaking axis directions.

It is important to note that the cancellation nodes are not due to the cross-over design, they are due to the vertically displaced drivers (the cross-over design controls where cancellation nodes occur, not that they occur). The greater the distance between two drivers, the more nodes we get. Actually, odd and even filters are very similar, the just change the direction of the main lobe. For an even order filter we would have the lobe facing straight on, as opposed to $-15°$.

So why do we care? Well the issue is made clearer by considering a speaker's directivity over an audience.

If we use an odd order Butterworth filter as we have suggested (gives us the flat crossover response, when time comes for the 1.7 kHz flute solo the everyone has been waiting for, the on-axis response sounds great, but due to the peaking axis lobe anybody at $-15°$ get flute blasted! Meanwhile, at the top and bottom the audience don't get to hear the solo at all (can you imagine the disappointment!).

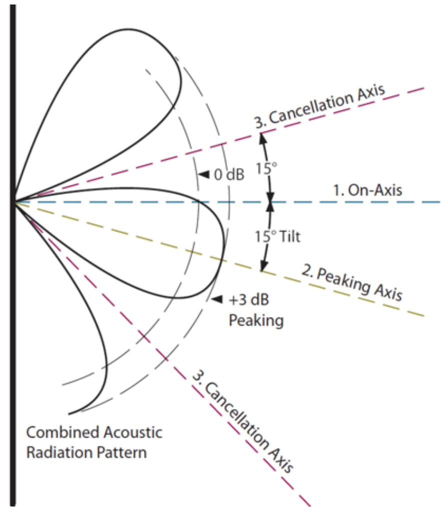


Figure 12.13: Vertical directivity due to non-collocated drivers

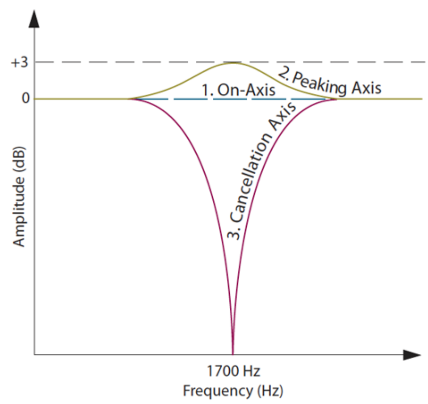


Figure 12.14: On-axis, peaking axis, and cancellation axis frequency response due to non-collocated drivers

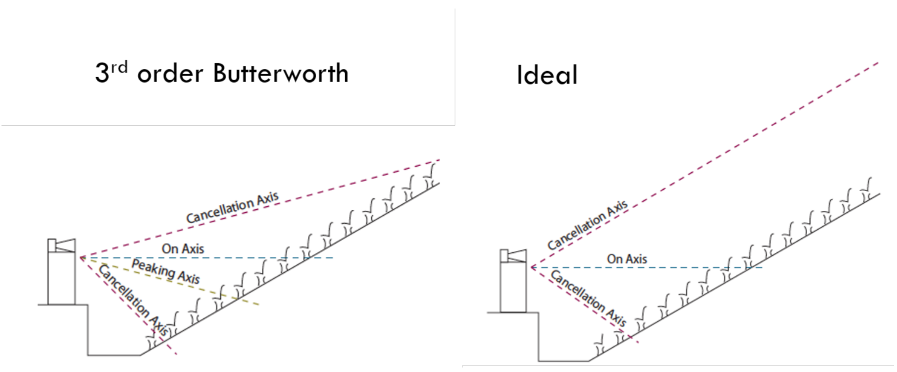


Figure 12.15: Speaker directivity over an audience.

Here is the problem. The Butterworth filter is not ideal as all orders have this problem, all that changes is the location of nulls and peaks. So what do we want instead? Well, for a start the main lobe should point directly forwards so that cancellation nulls are rotated away from the listeners (i.e. the drivers should be in phase). Also, the main lobe should not peak at +3dB (i.e. the filters gain should be 1).

How do we achieve this? Somehow we need to design a cross-over filter that gives us a gain of 1 whilst at the same time leaving both drivers in phase! This will be the topic of the next section.

13 Linkwitz-Riley Filters

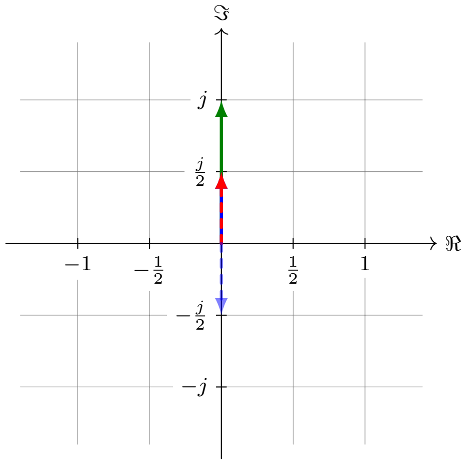


Figure 13.1: Low pass (blue), high pass (red), and combined (green) 2nd order responses of ideal filter in complex plane.

The issue we are trying to solve is the lobing effect. Lets think back to the 2nd order Butterworth filter's response. We had a $180°$ phase offset (see figure 12.5), but if we invert one of the drivers, they become in phase (see figure 12.6), in turn giving us a 3 dB boost. What if we could ensure that the drivers were in phase (so as to avoid the off-axis lobing), but control their gain so that when combined we get 0 dB, not 3 dB?

How would we do this? Rather than ensuring that the power at the cross-over was $1/2$ (so that amplitude was $1/\sqrt{2} = 0.707$), we should make it that the amplitude is $1/2$! We could do this by ensuring the power is $1/4$ rather than a $1/2$ (since $1/\sqrt{4} = 1/2$). This is equivalent to crossing over the filters at -6 dB, rather the -3 dB.

We can interpret this visually using figure 13.1. Previously in figure 12.6 we had the high pass and low pass transfer functions both equalling $\pm j/\sqrt{2}$ at the cross-over frequency. Instead, in figure 13.1 we have them equalling $\pm j/2$. Their summation then yields a magnitude of 1.

Here is an example of what we are hoping to achieve. By crossing the filters over at their -6 dB point the resulting level would be 1, and the drivers would remain in phase.

This would give us a main lobe at 0 degrees, as the drivers are in phase, and the magnitude of the peak would be 0dB. This is the idea behind the Linkwitz-Riley filter design.

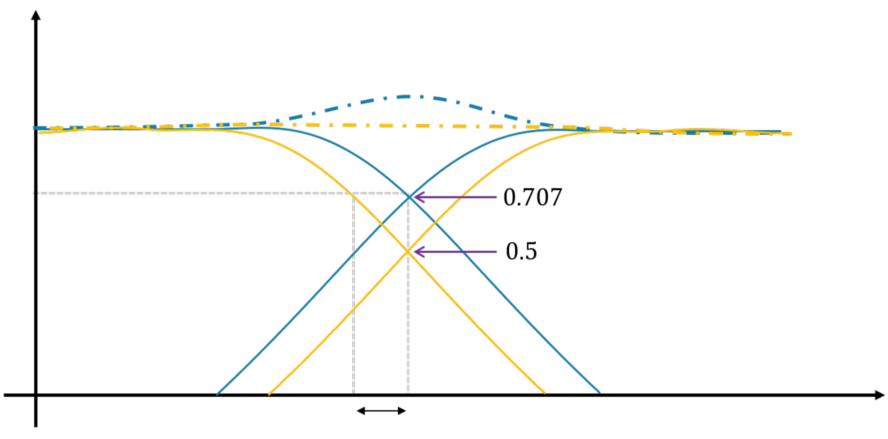


Figure 13.2: Filter cross-over at -3db vs.-6dB.

There are two ways of thinking of the Linkwitz-Riley filter:

1) One is as a new type of filter alignment. This time, we want amplitude, not power, to be $1/2$. We are also specifying a particular phase relationship, either

$\pm 90°$, (we can always invert one driver). We will look at this in a bit more detail shortly, but the idea is that we want the transfer function to be equal to $\pm j/2$ at ω_c. The task is then to find which component values L and C produces this.

2) The other way to think about it is to think of the Linkwitz-Riley filter as two Butterworth filters cascaded together. This was the original idea in the 70s. We will look at this a little later. For now lets go through an example.

Lets start with the transfer function we derived for our low pass 2nd order filter,

$$H(j\omega) = \frac{1}{\frac{j\omega L}{Z_{LS}} + (j\omega)^2 LC + 1} = \frac{1}{(1-\omega^2 LC) + \frac{j\omega L}{Z_{LS}}} \tag{13.1}$$

We are interested in the alignment where at ω_c we have,

$$H(j\omega_c) = \pm\frac{j}{2}. \tag{13.2}$$

This requires a $90°$ phase shift, i.e. the real part of the transfer function should be zero. This gives an equation we can use to solve for ω_c in terms of L and C,

$$1 - \omega_c^2 LC = 0 \rightarrow \omega_c^2 = \frac{1}{LC}. \tag{13.3}$$

Next we have an equation for its magnitude, which we set to $1/2$,

$$|H(j\omega_c)| = \frac{1}{2} = \frac{1}{\frac{j\omega L}{Z_{LS}}}. \tag{13.4}$$

This gives us a second equation.

Now we have a pair of simultaneous equations we can use to solve for L and C. From equation 13.4 we have that,

$$L = \frac{2Z_{LS}}{\omega_c}. \tag{13.5}$$

and from equations 13.3 and 13.5 we have that,

$$1 = \omega_c^2 CL \rightarrow C = \frac{1}{2Z_{LS}\omega_c} \tag{13.6}$$

Do these equations look familiar? Well they are quite similar to what we derived for the normal Butterworth alignment, except we have coefficients 2 and $1/2$ rather than $\sqrt{2}$ and $1/\sqrt{2}$.

Substituting these values into the low pass filter transfer function we get the following equation,

$$H(j\omega) = \frac{1}{\frac{j\omega L}{Z_{LS}} + (j\omega)^2 LC + 1} \rightarrow \frac{1}{1 + 2\left(\frac{j\omega}{\omega_c}\right) + \left(\frac{j\omega}{\omega_c}\right)^2}. \tag{13.7}$$

Again, this is very similar to the transfer function of the Butterworth alignment (see equation 11.19), except we have a factor of 2 in the denominator, not $\sqrt{2}$. Similarly for a high pass Linkwitz-Riley filter we have,

$$H(j\omega) = \frac{1}{\frac{1}{j\omega C Z_{LS}} + \frac{1}{(j\omega)^2 CL} + 1} \rightarrow \frac{1}{1 + 2\left(\frac{\omega_c}{j\omega}\right) + \left(\frac{\omega_c}{j\omega}\right)^2}. \tag{13.8}$$

Shown in figure 13.3 are the magnitude and phase responses of the low and high pass Linkwitz-Riley filters described above. Also shown is the combined response.

As expected we can see that the two filters cross-over at an amplitude of $1/2$. Also, their combined response is uniform across the entire frequency range, and they are in phase! This is exactly what we wanted.

By having the two filters/drivers in phase the main directivity lobe (i.e. the peaking axis) is directed straight forward. With the two drivers now having a gain of $1/2$ at the cross-over frequency, their combined response gives total gain of 1.

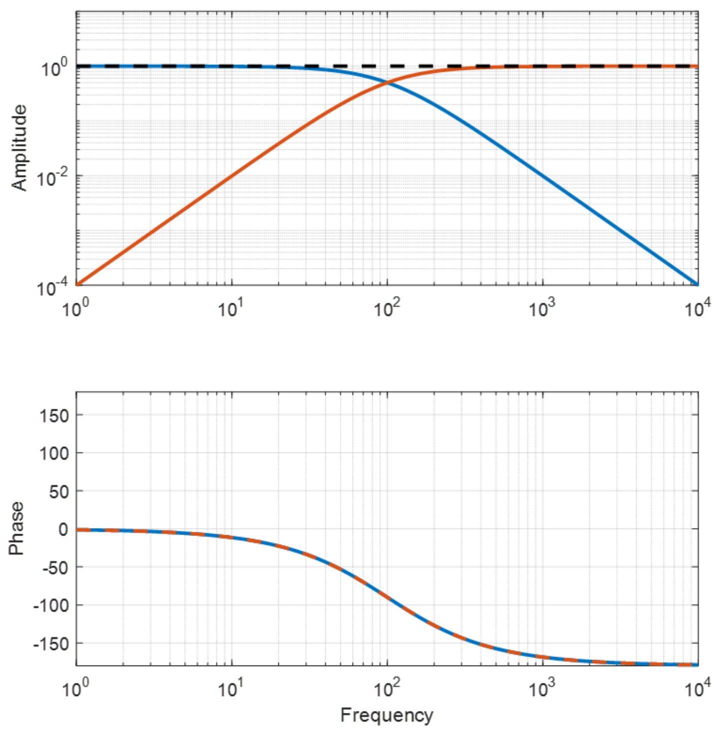


Figure 13.3: Magnitude and phase response of a low pass and high pass Linkwitz-Riley filters and their combined response.

In the above we consider the Linkwitz-Riley filter as a new type of alignment, and derived the L and C values to achieve it. The other way to think of a Linkwitz-Riley filter is as two Butterworth filters cascaded together. This was the original idea in the 70s. Shown in figure 13.4 in pink is a 2nd order filter Butterworth (-3dB cut off -12dB / oct roll off), in green is a third order Butterworth (-3dB cut off -18dB/oct roll off) and in blue is a 4th order Linkwitz-Riley filter (two cascaded 2nd order filter Butterworth). If we cascade two of the 2nd order Butterworths we can see how the gain at ω_c moves from -3dB to -6dB, and the rate goes from -12 to -24 dB/oct. You can also imagine how the phase changes. Even order Butterworths are always either in phase or 180 degree out of phase. Odd order Butterworths have $\pm 90°$ difference. If we cascade two of these they become in phase or $180°$ out of phase. Therefore cascading two will always yield the same result (in phase or 180 degree out).

So what's the catch? Well not much actually! We can summaries its properties as follows:

1) Always even order (made up of 2 Butterworth filters cascaded)

2) They produce a flat frequency response on axis

3) The main lobe is at zero degrees and is 0dB not $+3$dB

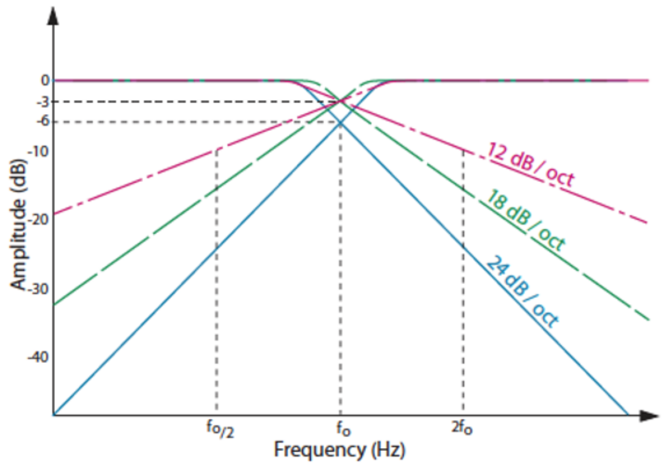


Figure 13.4: Linkwitz-Riley filter as a cascaded Butterworth filter

So what's not to like? Actually not much. Linkwitz-Riley's are probably the industry standard. In particular, 4th order systems seem to be very popular, but also 2nd order.

The main issue with using a Butterworth filter (or cascading them) is that a Butterworth filter has non-linear phase response. What does this mean? Just that the phase changes non-linearly with frequency.

Often in filter design it is interesting to look at the so called group delay. The group delay is the derivative of phase response with respect to frequency. It tells us the delay, in seconds, of different frequencies passing through the filter. With a linear phase response, the gradient is constant, thus all frequencies are delayed the same! So a non-linear phase response means that the group delay is not constant with frequency, i.e. some frequencies pass through quicker than others.

This non-linear phase behaviour can also be seen in the filter's step response in figure 13.5. We can see a slight overshoot when responding to a step response, which takes some time to damp down. Is this sort of artefact audible though? In studies it has been shown that under laboratory conditions you can detect the difference on non-musical tones, but in practical audio systems not so much, since the response in the order of 1 or 2 ms.

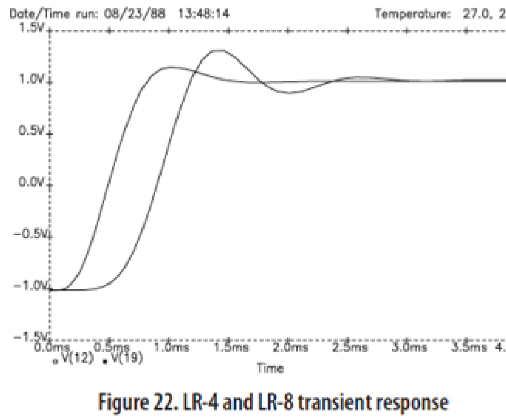


Figure 13.5: Step response of a Linkwitz-Riley filter.

14 Impedance Compensation

Up until now we have assumed that our loudspeaker impedance is constant and purely resistive, $Z_{LS} = R_{LS} = R_E$, where R_E is the drivers nominal DC resistance, despite knowing full well that it is far from constant. We saw earlier (see equation 8.29 and figure 8.11) that the electrical impedance of a loudspeaker driver is complex and frequency dependent. This means that our cross-over filter is 'looking into' a completely different circuit. This will change its properties, see for example figure 11.4.

It turns out that we can improve the performance of a cross-over by compensating for the loudspeaker's complex impedance, i.e. we can trick the cross-over into thinking that there is only a nominal resistance on its output. This is what we call impedance compensation. We saw a little while back that the cross-over response with the true and nominal loudspeaker impedance (see figure 11.4). We handwavingly described it as not too bad. But it is not perfect, and if we account for it we will get a better cross-over response. This is the sort of thing higher end systems will do.

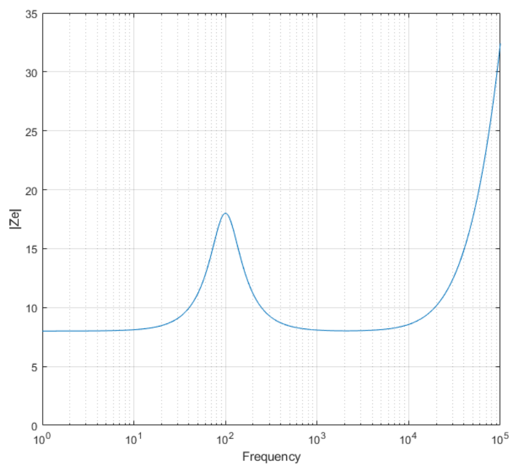


Figure 14.1: Electrical impedance of a moving coil loudspeaker in an infinite baffle.

The electrical impedance of a loudspeaker is made up of two key features. A low frequency peak due to the mechanical resonance of the driver (mass on a spring), followed by a gradual rise with high frequencies due to the inductance of the voice coil. These are the features we want to compensate for.

The trick is to add an extra impedance element into the filter network which when combined with the loudspeaker impedance, provides exactly the nominal DC resistance that the crossover was designed to work with. Adding the compensation impedance cancels out all the messy stuff that the loudspeaker creates.

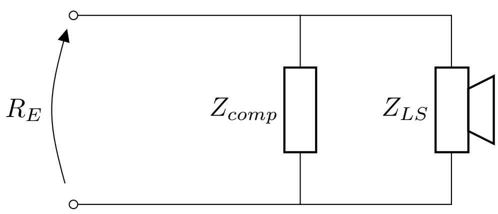


Figure 14.2: Impedance compensation circuit

Using the product over sum rule from figure 14.2 we have that,

$$R_E = \frac{Z_{comp} Z_{LS}}{Z_{comp} + Z_{LS}} \to Z_{comp} = \frac{Z_{LS} R_{LS}}{Z_{LS} - R_E}, \tag{14.1}$$

where Z_{comp} is the compensation impedance necessary to get a nominal resistance R_E across the terminals to the driver.

All we need is R_{LS} and Z_{LS}. We know R_E, its just the nominal DC resistance of the loudspeaker. What about Z_{LS}? Well we have already developed an equivalent circuit model for exactly this!

Shown in figure 14.3 is the equivalent circuit for a loudspeaker driver with all mechanical components referred to the electrical domain, excluding any acoustic loading.

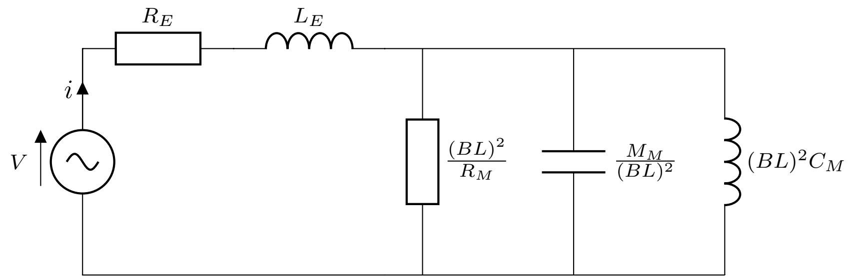


Figure 14.3: Equivalent circuit with all elements referred to the electrical domain.

Now all we need to do is work out the impedance of this circuit and stick it into the equation we formulated for Z_{comp}. However, it is easier to split the circuit into low and high frequency parts, where the low frequency part captures the resonance of the driver, and the high frequency part captures the increased impedance due to the voice coil inductance. Then we can compensate each driver separately, i.e. tweeter compensation for resonance and woofer compensation for coil inductance.

Here are the high and low frequency parts of the circuit. It has been split such that the low frequency part ignores the inductance, and the high frequency part ignores the mechanical resonance.

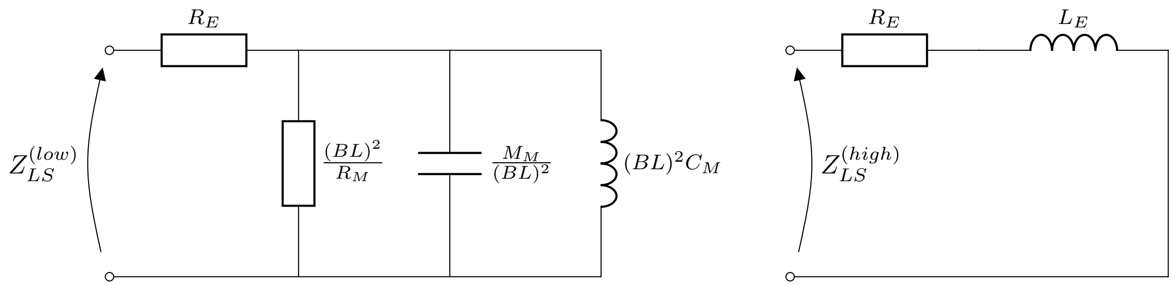


Figure 14.4: Low frequency (left) and high frequency (right) equivalent circuits for a loudspeaker driver in the electrical domain.

It is helpful to think of Z_{LS} as,

$$Z_{LS} = R_E + Z' \tag{14.2}$$

where Z' is some arbitrary impedance. This makes the maths easier! After substituting equation 14.2 into equation 14.1, we can rearrange as follows,

$$Z_{comp} = \frac{Z_{LS} R_E}{Z_{LS} - R_E} = \frac{(R_E + Z')R_E}{(R_E + Z') - R_E} = \frac{R_E^2 + Z' R_E}{Z'} = R_E + \frac{R_E^2}{Z'} \tag{14.3}$$

Now we have a new expression for Z_{comp} lets derive the high and low frequency compensation impedances.

14.0.1 High Frequency Compensation

It is straightforward from observation to see that Z' is simply the voice coil inductance.

$$Z_{LS} = R_E + Z' = R_E + j\omega L_E \tag{14.4}$$

Substituting this into Z_{LS} and this into Z_{comp} we arrive at the compensation impedance required for the voice coil inductance.

$$Z_{comp} = R_E + \frac{R_E^2}{Z'} = R_E + \frac{R_E^2}{j\omega L_E} \tag{14.5}$$

It is simply a series resistor and capacitor in parallel with the loudspeaker driver. The value of the capacitor is the voice coil inductance, divided by the DC resistance squared.

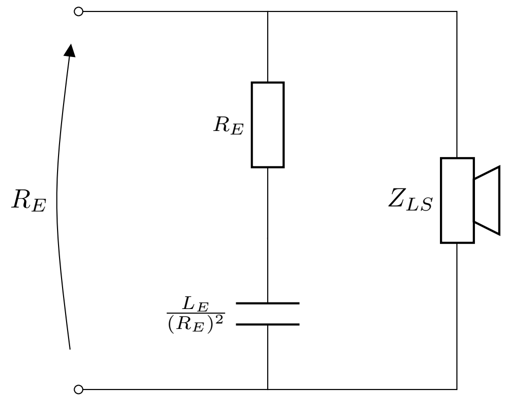


Figure 14.5: High frequency impedance compensation circuit

14.0.2 Low Frequency Compensation

Now for the low frequency compensation. We have our three mechanical elements in parallel. Using the inverse summation rule for parallel impedances we get the impedance Z',

$$Z' = \left(\frac{R_M}{(Bl)^2} + \frac{j\omega M_M}{(Bl)^2} + \frac{1}{(Bl)^2 j\omega C_M} \right)^{-1} \tag{14.6}$$

From inspection we can see that the $1/Z$ in the equation for Z_{comp} is simply the bracketed term above, i.e. the term being inverted.

$$Z_{comp} = R_E + \frac{R_E^2}{Z'} = R_E + R_E^2 \left(\frac{R_M}{(Bl)^2} + \frac{j\omega M_M}{(Bl)^2} + \frac{1}{(Bl)^2 j\omega C_M} \right) \tag{14.7}$$

The compensation impedance is now a simple sum of terms, i.e. the elements are in series! (still in parallel with the loudspeaker though). The corresponding values are shown in figure 14.6.

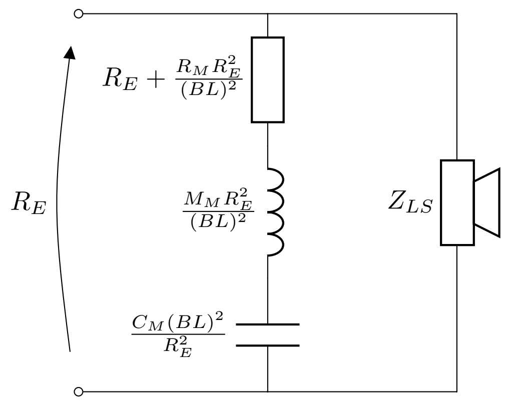


Figure 14.6: Low frequency impedance compensation circuit

14.0.3 Combined Impedance Compensation

The low and high frequency compensation networks derived above can be combined in parallel to provide broadband compensation for both mechanical and inductive features. The resulting circuit is shown in figure 14.7.

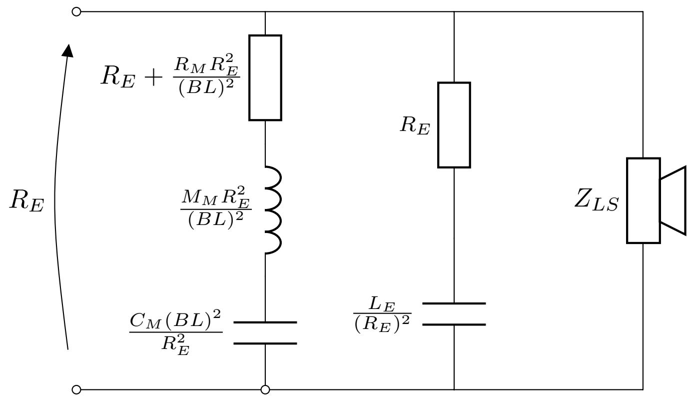


Figure 14.7: Combined impedance compensation circuit

Shown in figure 14.8 is an example of this sort of compensation network in action. We can see the high frequency compensation works well but it is more difficult to control the low frequency resonance. Never-the-less it does a good job reducing its influence.

This sort of compensation will make the cross-over more accurate, and isn't very expensive as it just requires a few extra passive components. In practice however, it is likely sufficient to just compensate bass driver for high frequencies, and perhaps the tweeter its mechanical resonance only.

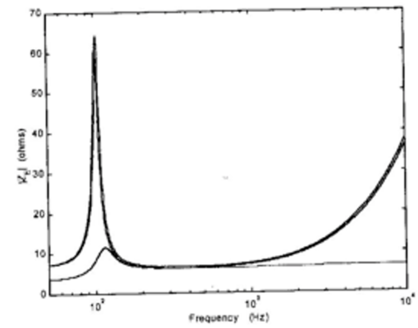


Figure 14.8: Impedance compensation example.

Part III

Loudspeaker Arrays

15 Introduction

In the first semester we looked at standard single driver loudspeakers. We used a rigid piston model to approximate their radiation and we saw that at low frequencies they behave more or less omni directionally. As frequency increases, the directivity of a rigid piston gets more beam like.

Although this sort of behaviour is typical for standard loudspeakers, often we want a bit more control over the directivity. This sort of control however, doesn't come cheap...

Shown in figure 15.1 is a B&O speaker, released about 10 years ago, nicknamed the darlek (BeoLab 5 Active Loudspeakers). This odd looking loudspeaker has been designed to give a uniform 180 degree dispersion, i.e. an even directivity at all frequencies. This means it does't matter where you sit still get the same sound. The only problem is that it costs about £20,000.

Figure 15.1: BeoLab 5

Nowadays, more HiFi loudspeakers are considering the effect of the room, and their own placement, typically by adding some directivity control. Although relatively new on the HiFi scene, this sort of control has been an integral part of PA design for quite some time.

The Beolab 5 was designed to disperse sound evenly. Another, this time even more expensive (£60,000), B&O loudspeaker (the Beolab 90, see figure 15.2) is designed to offer a variety of controllable directivity patterns. Sometimes you don't want 180° dispersion. Music is mixed in a studio, i.e. a space designed so that reflections are diffuse. This sort of diffuse field will not be achieved in your home.

Figure 15.2: BeoLab 5

So how do we achieve a more controlled directivity? The trick is to use multiple loudspeaker drivers. This will allow us to generate directional responses, which can be controlled, for example using beam steering. This was you can take into account the room within the design.

15.1 The Line Array

The first directional loudspeaker we will consider is the line array. I'm sure you will recognise this type of loudspeaker design (see figure 15.3). Its certainly not something you would have set up in your living room, but it is pretty much the defacto standard for large area/out door performances, or any other large scale PA system. We will begin our foray into loudspeaker arrays by looking at how to model the sound radiation from line array loudspeakers. Hopefully some of the maths we cover will already be familiar to you from your other modules.

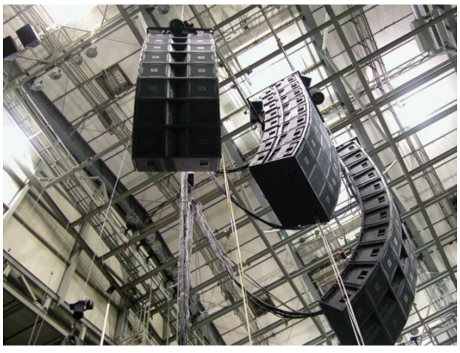


Figure 15.3: Example line array.

Lets start with a bit of historical perspective. The earliest known research into line array is thought to be the work for Harry F. Olson, a famous American electrical and acoustical engineer, in 1957. Olson developed the column speaker, a series of vertically aligned drivers in a single enclosure. This particular design achieved a wide horizontal and narrow vertical directivity pattern. It became a popular choice for speech reinforcement, particularly in reverberant spaces. Its focussed directivity minimised unnecessary excitation of the reverberant field, thus improving the direct-to-diffuse ratio desirable for clarity and intelligibility.

It wasn't really until early 90s that line arrays were revisited and used for full range PA systems. One key problem was that they only worked over a narrow frequency band. At high frequencies you can't locate drivers close enough. This leads to constructive and destructive interference with multiple hard to control side-lobes. Although one solution would be to reduce the loudspeaker size to space them closer together, we need large loudspeakers for low frequency radiation!

In 92, L-acoustics made a break through by developing a waveguide system which restricted the directivity at HF. The idea was that rather than using constructive and destructive interference, horns were use to provide the required directionality. The L-acoustics V-DOSC system was a game changer in sound reinforcement. Their success is evident from their 20 year production run, from 1992 until quite recently. In fact they were the main loudspeaker system for the London 2012 Olympic Stadium.

So back to the problem at hand... How do we model, mathematically, the sound radiation from a line array? Well first of all, what is a line array? In its simplest form, it is a number of ideal acoustic sources arranged in a line. It turns out that a very similar problem was tackled was back in the 17th century!

15.2 Huygen's Principle

Figure 15.4: Christian Huygens, 1629-1695

The whole point of any loudspeaker system is to recreate a captured sound field, as closely and as realistically as possible. It turns out a 17th century chap called Christian Huygens (see figure 15.4) was well ahead of his time. He argued that any wave front could be modelled as a series of wavelets, in this case the wavelets are circular waves, emanating from ideal monopole sources.

Simply put, if you place a series of monopoles over a wave front they will interfere with one another in such a way that they recreate that wave front. This means that we can recreate a wave front from any location! I.e. we don't have to go back and recreate the original source. Figure 15.5 shows the principle being used to recreate both circular and plane waves.

This is the fundamental principle that underpins the use of line array loudspeakers (also other more complex loudspeaker systems, for example Wave Field Synthesis arrangements)

15.3 Vertical and Horizontal Coverage

Before we get into the mathematical modelling, lets consider more generally the design of line arrays. I'm sure you have noticed that the systems used for sound reinforcement outdoors and in large venues always end up curved into a "J" shape. But why?

Well first of all, we want a line array to prevent spill, i.e. minimise radiation away from audience (for example to avoid annoying residents nearby a festival). Second, we want to curve the wave front downwards so that nearby listeners can hear what's going on (otherwise we would just project it over the top of them!).

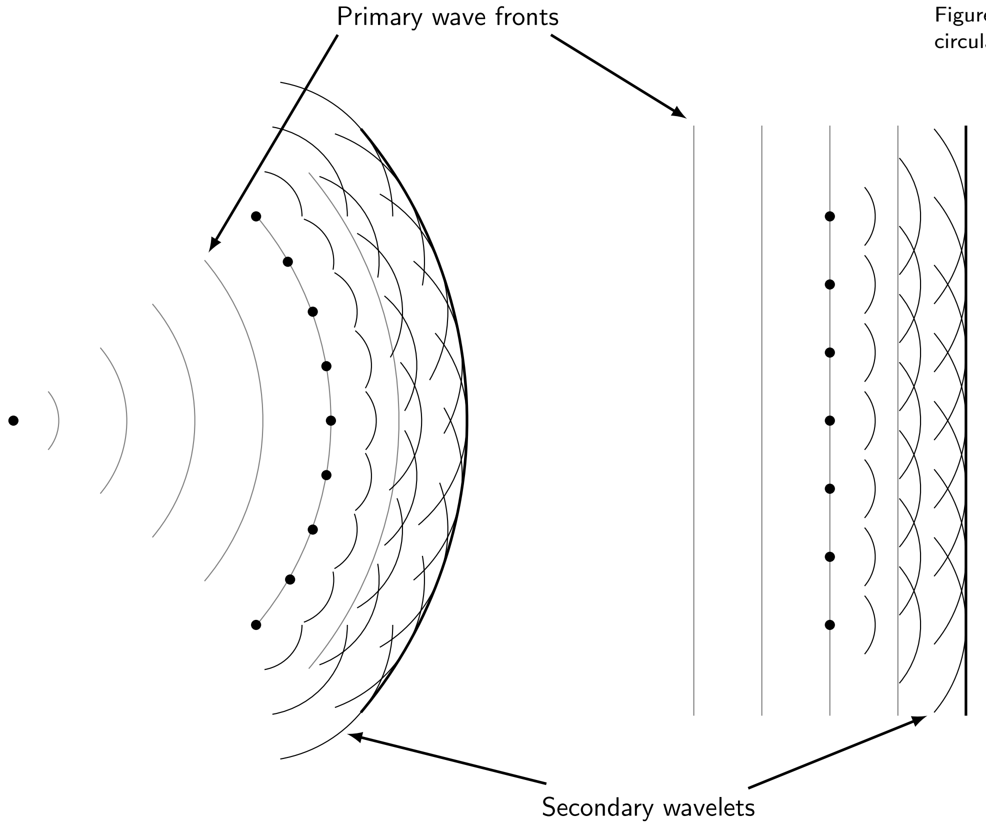


Figure 15.5: Huygen's principle used to recreate circular and plane wave fronts.

Ideally, we want equal sound intensity over the entire audience. Now remember, intensity is proportional to $1/r^2$. So far away we need to have lots of speakers contributing. These speakers interfere constructively to deliver more intensity at a distance. This is done by forming a narrow focused beam. To compensate for being close, less speakers contribute, so we get a wider beam with lower intensity, as in figure 15.6.

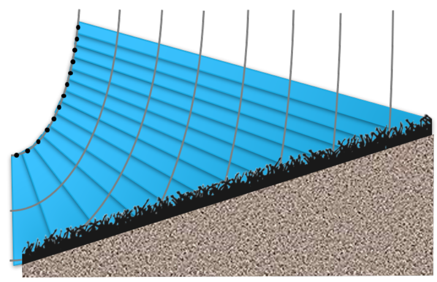


Figure 15.6: Vertical coverage from line array

Now what about horizontal coverage (figure 15.7)? This is also important. We want a wide directivity across all frequencies. The loudspeaker cabinets are often designed to try and achieve this. Shown in figure 15.8 is an example. We have a narrow aperture for small wavelengths (high frequencies) with cabinet aided diffraction to give wide dispersion. Often systems will use horn loading to give even dispersion at multiple frequencies.

At low frequencies widely spaced speakers are unavoidable. This leads to wide dispersion at larger wavelengths. At very low frequencies directivity is more difficult to control and requires other solutions, which we will cover latex in the notes.

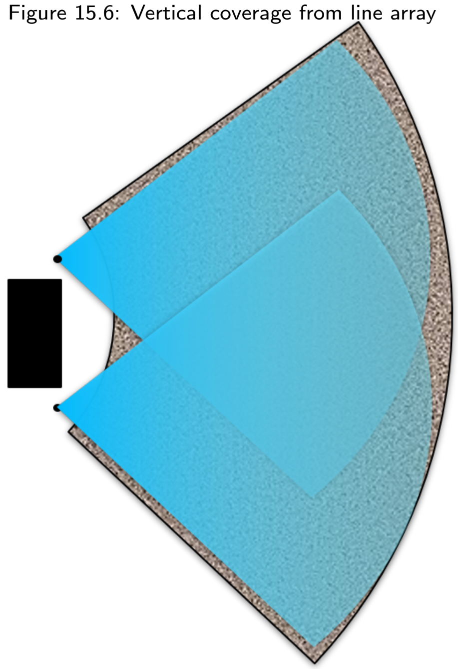


Figure 15.7: Horizontal coverage from two speakers

16 Line Array - Mathematical Model

Okay, lets start building our line array model. To start with let us assume that our individual acoustic sources can be treated as monopoles (i.e. considering low frequencies). Our line array can thus be modelled as a series of monopoles in a row.

So what is a monopole? We covered this earlier when looking at radiation models for loudspeakers. A monopole is an idealised acoustic source whose radiation is omni-directional. We can think of it as an infinitely small pulsating sphere.

Suppose we have N monopoles, arranged over a line of length L, spaced d meters apart from each other (see figure 16.1). Each monopole is driven by a complex signal A_n. Also, assume that a listener is positioned at some large distance $r >> L$ from the array. Recall that each monopole radiates spherical

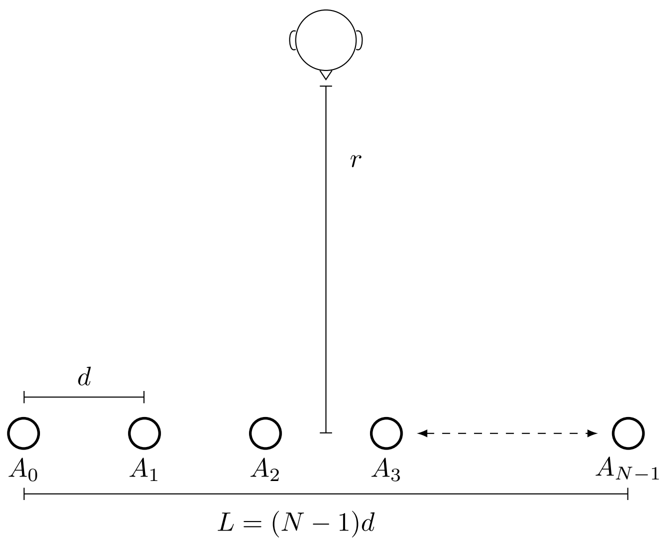


Figure 16.1: Line array model

waves, and that at any point in space these waves will add together. It is clear from figure 16.2 that if the listener is positioned sufficiently far away (i.e. if $r >> L$), the summed response of each spherical wave will approximate a plane wave. This corresponds to the so called 'far field approximation'. Providing that our listener is located in the far field we can think of our line array as simply radiating plane waves. In other words, we can assume that rays from sources remain parallel to one another. Also note that based on the far field assumption, the difference in level between each loudspeaker's contribution may be considered negligible, since plane waves do not attenuate with distance. Now we can think about what happens at different angles. As you rotate around in the far field we still have plane waves. Under the far field assumption all of these parallel rays

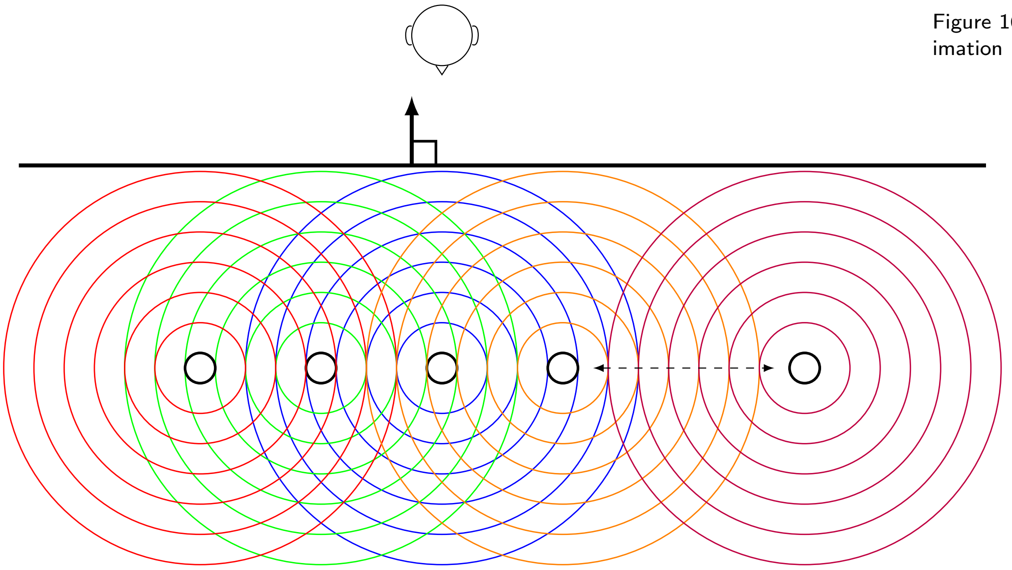


Figure 16.2: Line array model - far field approximation

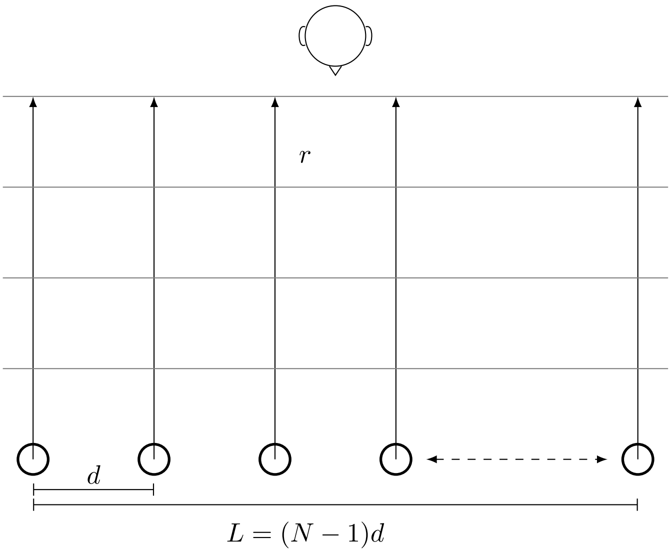


Figure 16.3: Line array model - far field approximation

have the same angle θ to the receiver. Clearly this assumption breaks down as you get closer to the array, i.e. each angle will be different for a curved wave-front.

If we do a little geometry we can get a bit further. What we want is an expression for the path length difference between each source and the receiver. So if the difference between r_0 and r_1 is Δr, then clearly the difference between r_0 and r_2 is $2\Delta r$ and so on. So we want an expression for Δr in terms of the angle θ.

Shown in figure 16.5 are two elements of our line array. From inspection we have,

$$\sin\theta = \frac{\Delta r}{d} \rightarrow \Delta r = d\sin\theta. \tag{16.1}$$

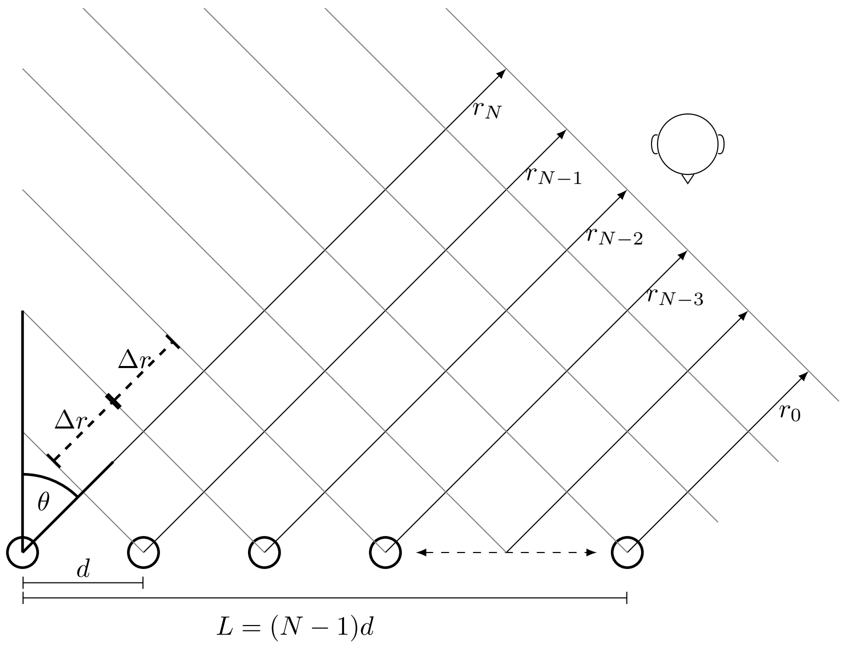


Figure 16.4: Line array model - far field rotation

Now lets put out line array model together. First recall the equation for monopole radiation,

$$p(r,\omega) = \frac{A}{r} e^{j(\omega t - kr)}. \tag{16.2}$$

Now separate the time and space variables in the exponential,

$$p(r,\omega) = \frac{A}{r} e^{j\omega t} e^{jkr} \tag{16.3}$$

Now we can consider the nth monopoles radiation by extending the path length,

$$p(r,\omega)_n = \frac{A}{r} e^{j\omega t} e^{jk(r+n\Delta r)}. \tag{16.4}$$

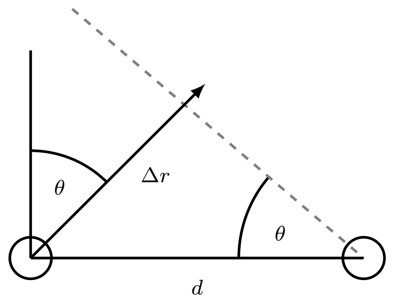


Figure 16.5: Path length difference

Note that we have assumed that the attenuation due to distance is unchanged by the additional path length. This is part of our far field assumption. Now lets regroup the complex exponential and define a new complex amplitude coefficient that contains the coefficient A, the distance attenuation, and the original time/space dependency,

$$p(r,\omega)_n = \frac{A}{r} e^{j(\omega t - kr)} e^{jkn\Delta r} = A_n e^{jkn\Delta r} \tag{16.5}$$

where $A_n = \frac{A}{r} e^{j(\omega t - kr)}$ This new exponential term $e^{jkn\Delta r}$ describes the change in phase due to the additional path length $n\Delta r$.

We now have a expression for the radiation of the nth monopole, lets add together the contribution of each monopole

$$p(r,\omega)_T = A_0 + A_1 e^{jk\Delta r} + A_2 e^{jk2\Delta r} + \cdots + A_{N-1} e^{jk(N-1)\Delta r} \tag{16.6}$$

or in summation notation,

$$p_T(r,\omega) = \sum_{n=0}^{N-1} A_n e^{jkn\Delta r} \tag{16.7}$$

This gives us the total radiation pressure due to our line array. Now let us substitute in our trigonometric identity for the path length difference, this gives us the radiated pressure as a function of angle,

$$p_T(\theta, \omega) = \sum_{n=0}^{N-1} A_n e^{jknd\sin\theta}. \tag{16.8}$$

Now Lets define a new variable, $\Omega(\theta) = kd\sin\theta$. Remember, k is the wave length, d is the loudspeaker spacing, and θ is the listener angle. Substitution into equation 16.8 yields an equation that should look somewhat familiar...

$$p_T(\Omega) = \sum_{n=0}^{N-1} A_n e^{jn\Omega(\theta)}. \tag{16.9}$$

It is the same form as the discrete Fourier transform of $(A_1, A_2, \cdots, A_N)$! This is a really interesting result. We have derived an expression for our line array radiation that looks just like the Fourier transform of its amplitude coefficients!

Before we take this useful fact any further, lets revisit the Fourier transform.

16.1 Fourier Transform

So what do you remember about the Fourier transform? The Fourier transform is an integral transformation of the form,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt, \tag{16.10}$$

where $x(t)$ is an arbitrary time varying signal, and $e^{j\omega t}$ is a complex exponential (i.e. a sinusoid of frequency ω with some arbitrary phase). Note that here t is a continuous variable, i.e. it takes on all real values, hence equation 16.10 is a *continuous* Fourier transform.

The discrete Fourier transform looks pretty similar though... We multiply each value in a discrete array $x(n)$ by a complex exponential, and sum over all values,

$$X(\omega) = \sum_{-\infty}^{\infty} x(n)e^{j\omega n}. \tag{16.11}$$

This complex exponential differs from the continuous Fourier Transform in that the continuous time variable is replaced by the discrete counter n.

So the Fourier transform is an integral transformation that relates two equivalent descriptions of a signal. Take for example the time varying signal $x(t)$ (this could be for example an acoustic pressure measured by a microphone). This time domain signal is related by the Fourier transform to its frequency domain representation $X(\omega)$. In the time domain, the signal is represented by a complex wave form that varies with time. In the frequency domain the signal is represented by an infinite summation of simple sine waves, which are periodic in time. These two descriptions of the signal are entirely analogous. So the time domain and frequency domain representations of a signal are related by the Fourier transform (and visa versa by the inverse Fourier transform). What other domains are there?

The time domain is used to describe information that changes with time. But information can also change with position, for example at any particular instant an acoustic wave has some spatial variation. So we also have spatial domain. Like

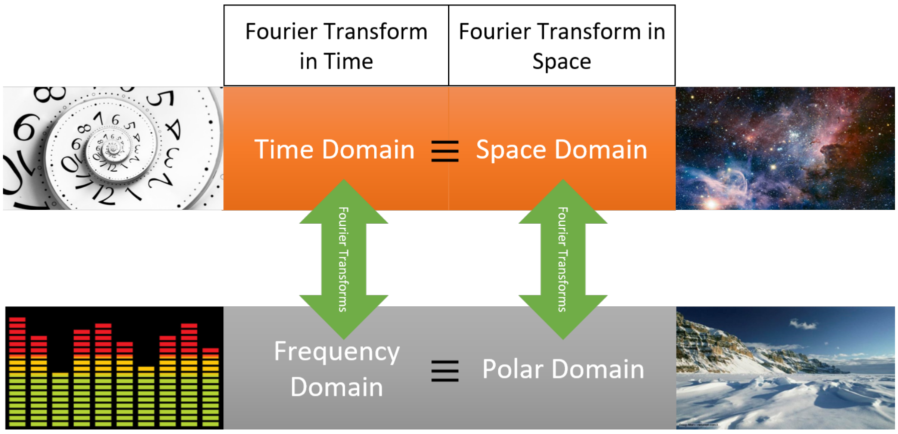


Figure 16.6: Fourier transforms - time, frequency, space and polar domains.

the time domain, the spatial domain also has a second equivalent representation, related through the Fourier transform. We call it the polar domain.

So the Fourier transform of a time domain signals gives a frequency domain response. Similarly, the Fourier transform of a spatial domain signal gives a polar domain response.

16.1.1 Useful Fourier Properties

So the big advantage of this Fourier based approach to line array directivity is that it opens up bunch of useful tools based on the properties of the Fourier transform. Here are some of the use Fourier transform properties that we will be using later on:

Linearity - Firstly if we add stuff together, in time or spatial domain, or scale it by a constant factor, it scales by the same amount in the transformed domain.

$$\mathsf{FFT}[ax(t) + by(t)] = a\mathsf{FFT}[x(t)] + \mathsf{FFT}[y(t)] \tag{16.12}$$

Shift theorem - In the time domain this is the theory that says if we want to delay to a signal, in the frequency domain this is equivalent to multiplying by a complex exponential, where the phase shift applied increases with frequency ($\omega_0 t$), i.e. it is a linear phase change.

$$\mathsf{FFT}[x(t - t_0)] = e^{-j\omega t_0}\mathsf{FFT}[x(t)] \tag{16.13}$$

Convolution - Multiplication in frequency domain is convolution in the time domain, and vice verse.

$$x(t) \otimes y(t) = \mathsf{FFT}[X(\omega) \times Y(\omega)] \tag{16.14}$$

$$x(t) \times y(t) = \mathsf{FFT}[X(\omega) \otimes Y(\omega)] \tag{16.15}$$

Keep these properties in mind as we continue talking about our line array model.

16.2 Fourier Based Polar Patterns

Now recall our line array model from equation 16.9,

$$p_T(\Omega) = \sum_{n=0}^{N-1} A_n e^{jn\Omega(\theta)}. \tag{16.16}$$

Note that this equation bares a considerable resemblance to the discrete Fourier transform in equation 16.11. Instead of a transformation between the time domain and the frequency domain, $t \to \omega$, we have a transformation between the spatial domain and the polar domain $n \to \Omega$. Remember, here n corresponds to the index of our monopole sources. Since these monopoles are evenly spaced along a line, we can think of n as being equivalent to position, $n \to x$. Now what about the summation?

The discrete Fourier transform involves an infinite summation from $-\infty$ to ∞, our line array model however involves a finite summation... No problem. Lets imagine that our line array does in fact extend from $-\infty$ to ∞, but only the 0 to $N-1$ coefficients are non zero! This essentially truncates the infinite summation to that of our line array model. So we can interpret our line array model as the discrete Fourier transform of an infinite array of amplitude coefficients A_n, only a finite number of which are non zero.

This is a big result. It suggests that if we have number of sources sitting on surface each radiating with a complex amplitude, in the *far field* the angular directivity (i.e. the polar response, the radiated pressure with respect to angle) can be obtained by taking the discrete Fourier transform of the amplitude coefficient pattern over the line array. This is an amazingly powerful result. It will let us apply all of the neat tricks that come with Fourier transforms, e.g. convolution, shift theorem, etc.

What's more interesting is that the above doesn't just apply to sound but also to other wave based phenomena, such as light, e.g. x-rays. When you do x-ray imaging you observe a far field polar pattern. By finding the inverse Fourier transform, we are able to determine an array of amplitudes at the source location.

Now back to our line array. Suppose we decide to drive each element of our line array with the same signal. Consequently, each monopole will have the same amplitude. If we were to draw the amplitudes they would look like figure 16.7. Now remember that the polar pattern is related to the amplitude coefficients by the DFT. So what is the DFT of this rectangular pulse type of signal? It is a sinc function! So what have we just concluded? By having an array of omnidirectional sources in a line, we obtain something that is not omnidirectional. It has a polar pattern, that will look like a sinc function. I.e. it will have one main lobe on axis and a series of minor lobes at other angles.

Lets look at an example. Lets consider an array of 11 monopoles spaced evenly over 1 meter, radiating at 1kHz. Shown in figure 16.8 and 16.9 are the radiated pressures, as a function of $\Omega = kd\sin\theta$, at distances of 1m and 10m, respectively. Two plots are given in each figure, the exact response and the far field approximation. The exact response was obtained adding together the response of each monopole, computing individually their distances (i.e. we do not assume plane wave propagation). The far field response was obtained using equation 16.9 by assuming plane wave propagation (i.e. rays from sources remain parallel). At 1m away from the array, the far field approximation is clearly in

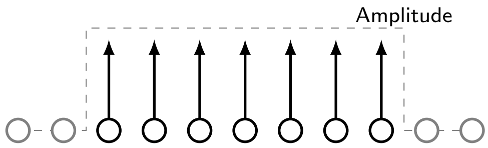


Figure 16.7: Infinite line array

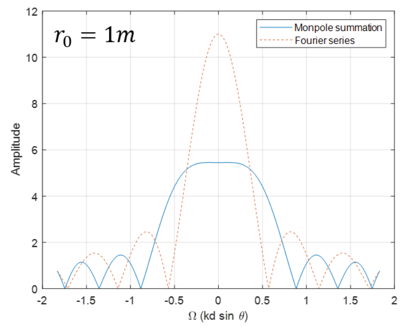


Figure 16.8: Line array (11 monopoles spaced evenly over 1 meter) radiation at 1 kHz at a distance of 1m

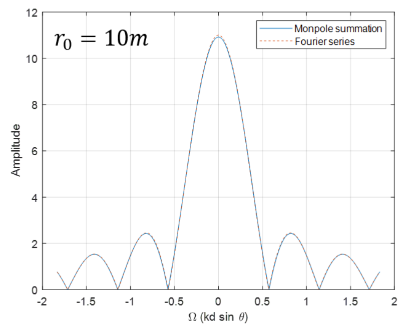


Figure 16.9: Line array (11 monopoles spaced evenly over 1 meter) radiation at 1 kHz at a distance of 10m

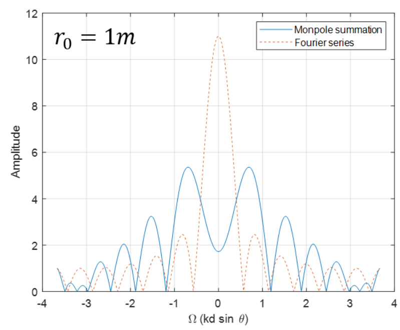
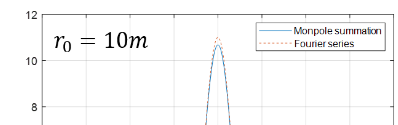


Figure 16.10: Line array (11 monopoles spaced evenly over 1 meter) radiation at 2 kHz at a distance of 1m



poor agreement with the exact solution. Whilst at 10m away we get excellent agreement. This demonstrates that our simplified (far field) model provides a reasonable approximation in the far field!

Shown in figure 16.10 and 16.11 are the radiated pressures of the same line array, instead at a frequency of 2 kHz. As before, in the far field our simplified model does a really good job at predicting the radiated pressure, compared to the exact solution. Also, notice that the main beam is now narrower, and we can see a greater number of minor lobes.

An important point needs to be made here. Although Ω can in theory vary over an infinite range, according to the geometrical set up of our problem ($\Omega = kd\sin\theta$), it can not exceed $\pm kd$, $-kd \leq \Omega \leq kd$. This is due to the $\sin\theta$ term, which has a maximum/minimum value of ± 1 when $\theta = \pm 90°$. Clearly, once we exceed $\pm 90°$ we will start to see a repetition of the directivity pattern shown in figure 16.8-16.11, i.e. we will begin to circle behind the array. The region of admissible Ω values is called the 'visible region'. Its limits are dictated by $\pm kd$, i.e. the monopole spacing, and the radiating frequency. For the example line array shown above, at 1 kHz $kd = 2\pi 1000 \times d/c = 1.8318$, and at 2 kHz $kd = 3.6637$. These values correspond to the $90°$ limits ($\sin\theta = \pm 1$) in figures 16.9 and 16.11, respectively.

If either the spacing d or frequency ω are increased, the visible region will expand. Consequently, the main on-axis lobe will become more narrow, and a greater number of minor lobes will be seen (see for example the difference between figures 16.9 and 16.11). Does this sort of behaviour remind you of anything? Rigid piston radiation! Except with the piston radiation we had a Bessel function directivity. Here we have a sinc function directivity.

16.2.1 Polar Periodicity

Lets look in a bit more detail at this repetition/periodicity of the polar response. This polar periodicity is not a feature unique to our line array model, instead it is a property of the discrete Fourier transform, and similar effects appear in host of other areas (e.g. aliasing in digital to analogue conversion!). To really appreciate some of the peculiarities of line array directivity, and its discrete Fourier transform interpretation, it is worth us revisiting Fourier transforms again.

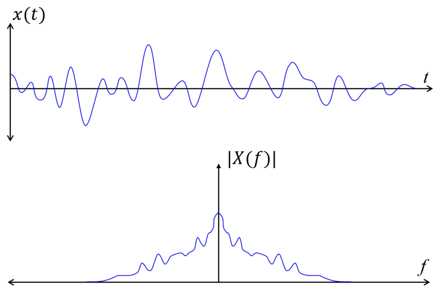


Figure 16.12: Continuous signal and its Fourier transform.

Fourier transforms are a bit of a mine field. There are a few different types. Lets start by assuming we have a continuous time domain signal $x(t)$, which we will further assume is infinite in length (see figure 16.12 top).

In this case we can happily apply the continuous Fourier transform and we will get an analytical solution, corresponding to the signals spectrum (see figure 16.12 bottom). It is important to remember that when applying the Fourier transform we integrate from $-\infty$ to ∞, and so we get both positive and negative frequencies (just like we have positive and negative angles when looking at directivity).

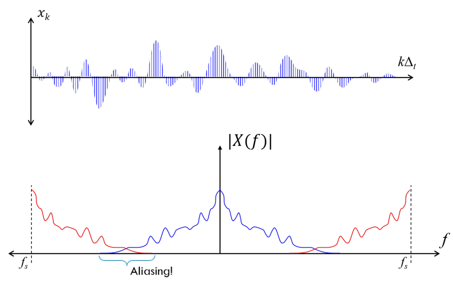


Figure 16.13: Discretised signal and its discrete Fourier transform.

Now suppose we discretise the time domain signal by sampling it. What will happen to its Fourier transform? If we sample a time domain signal we get periodicities in the frequency domain!

We can think of the process of sampling as multiplying our continuous signal with a pulse train whose rate is f_s (i.e. the sample rate). According to the convolution rule for Fourier transforms, this is equivalent to convolving the frequency response of the signal, with that of the pulse train. So what is the Fourier transform of the pulse train? Well its another pulse train, this time centred at integer multiples of f_s.

Convolution of these two frequency domain responses effectively replicates the signal spectrum around each pulse, at integer multiples of f_s. This leads to a periodicity in the frequency domain (see figure 16.13).

There is one important issue however, and that is if the signal moves faster than the pulse train. This introduces an ambiguity because we don't know what happens between the samples, i.e. has it moved gradually from one sample to the next? Or made some rapid transition?

Rapid transitions lead to what is known as aliasing. This is where the high frequency components of the repeated spectrum begin to overlap with the high frequency components of the initial spectrum. In Digital Signal Processing we normally apply very high order filters just before the Nyquist frequency ($f_s/2$) so as to avoid this effect. It is important to understand that the exact same problem arises in array directivity. What is the consequence? We get multiple main beams (this is very bad for our directivity)!

Lets look at an example. Shown in figure 16.14 is the radiated pressure from the say array consider before, this time at a frequency of 4 kHz, corresponding to $kd = 7.3273$. The effect of aliasing is quite clear here. We have introduced two major slide lobes at approx. $kd = 6.2$. So what causes this aliasing? It starts

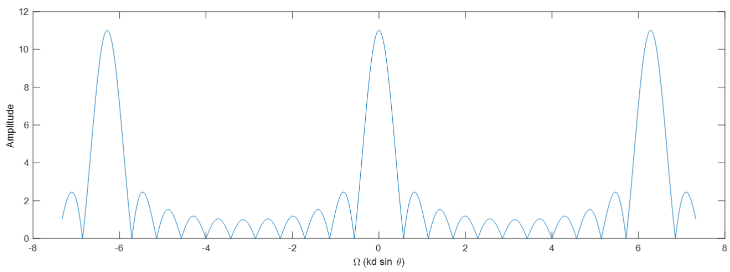


Figure 16.14: Line array (11 monopoles spaced evenly over 1 meter) radiation at 4 kHz at a distance of 10m

when *half the radiating wave length* ($\lambda/2$) is equal to the monopole spacing (d), $\lambda/2 = d$.

The example array considered has a spacing, $d = 0.1m$. A half wave length of $\lambda = 2d = 0.2 = c/f$, corresponds to a frequency of $f = 1715$ Hz. With a spacing of $d = 0.1$ we have $kd = 3.1416$. From figure 16.15 we can see that it is at this value of $\Omega = kd \sin\theta$ that our directivity begins to repeat. That is to say, taking $\theta = \pm 90$ (i.e. considering the off-axis response), after $f = 1715$ our directivity we begin to repeat itself. This corresponds to the Nyquist limit! At this frequency each source interferes destructively with the following on (since they are 180 degrees out of phase), causing a complete cancellation. Since we have modelled an odd number of monopoles, the final result is the contribution from just one single monopole (the rest have cancelled each other out). And so the amplitude is rather small. Had we modelled an even number of monopoles, the radiated pressure would be 0 at 90 degrees. Lets now consider what happens when the *whole radiating wave length* is equal to the monopole spacing, $\lambda = d$. This corresponds to a frequency of $f = 3430$ Hz and a $kd = 6.2832$. At this point that spacing between the monopoles is exactly the same as 1 wave length. This means that when looking at $\theta = \pm 90^\circ$ (or on axis), all of the monopoles interfere constructively. Note that this occurs at what is equivalent to the 'sample

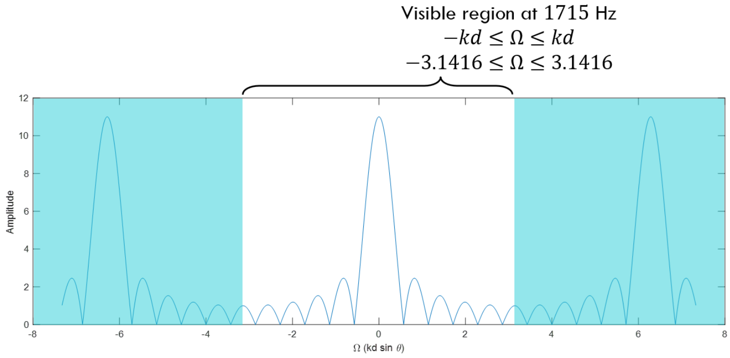


Figure 16.15: Line array (11 monopoles spaced evenly over 1 meter) radiation at 1.715 kHz at a distance of 10m

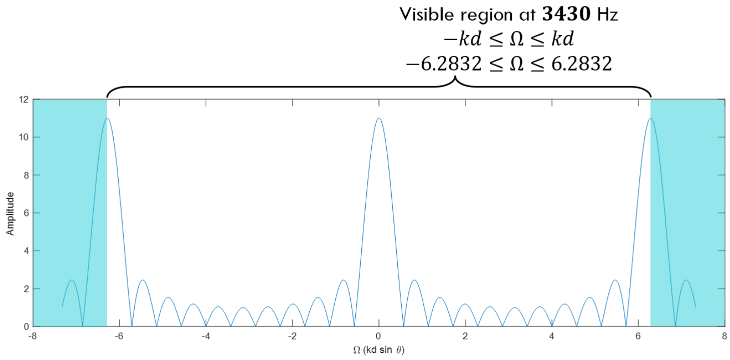


Figure 16.16: Line array (11 monopoles spaced evenly over 1 meter) radiation at 4 kHz at a distance of 10m

frequency'.

Now what about the smaller lobes? And all of the anti-lobes? Lets first think about what has to happen for us to get complete cancellation at 90 degree? We have already discussed that this would happen with an even number of monopoles if the spacing was half a wave length, but when else? When the wave length is equal to the length of the array then each monopole will cancel not with the following monopole, but a later monopole whose is spaced half a wave length away plus d. As this happens for each monopole, we get a complete cancellation at 90 degrees. This is easier to visualise with an even number of sources, as in figure 16.17. Each monopole is exactly $180°$ out of phase with another, so we get a complete cancellation at 90 degrees. This is exactly what happens at the first anti-node.

So at $90°$ we get a 0 response whenever,

$$\lambda = \frac{L+d}{n} = \frac{Nd}{n} \rightarrow k_0 = \frac{2\pi}{\lambda} = \frac{2\pi n}{Nd} \rightarrow (kd)_0 = \frac{2\pi n}{N} \tag{16.17}$$

This corresponds to a frequency of,

$$f = \frac{c}{nNd} \tag{16.18}$$

where $n = 1, 2, 3, \cdots$. For the example array considered above (where $N = 11$, $d = 0.1$) we calculate the first minimum to be at $f = 343$ Hz, for which $kd = 0.628$, which is in agreement with figure 16.14.

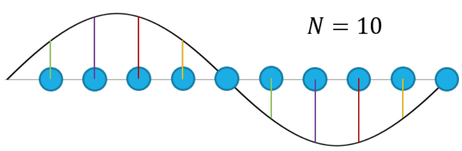


Figure 16.17: Destructive interference between 10 monopoles of an array.

As you might expect, the same thing happens once we have two wave lengths across the length of the array! And so on.

Now what about the minor lobes? These are less interesting but we might as well look at them whilst we are here!

If we have a full wave length over the line array, then each monopole will be cancelled by another which is positioned half a wave length away. This causes a complete destructive interference. Now what if we have slightly more than a full wave length across the array? Then those monopole which are within the wave length will cancel with one another, but those outside this wave length wont. They will interfere constructively.

As the wave length gets longer more of these extra monopoles (which aren't cancelled) get introduced, until we reach one and a half wave lengths. Clearly any further increase and the extra monopoles will begin to interfere destructively again. This will continue until we have two waves lengths across the array, which will give us complete cancellation again.

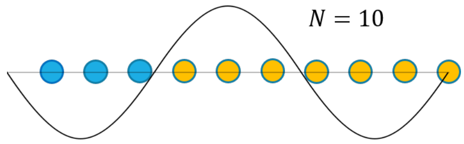


Figure 16.18: Destructive interference between 7 monopoles of an array.

For a minor lobe to occur at $\pm 90°$ we need the the length of the array to equal n and a half wave lengths, where n is some integer multiple,

$$\left(n+\frac{1}{2}\right)\lambda = L = (N-1)d \rightarrow \lambda = \frac{(N-1)d}{n+\frac{1}{2}}. \tag{16.19}$$

This corresponds to a frequency of,

$$f = \frac{c\left(n+\frac{1}{2}\right)}{(N-1)d}. \tag{16.20}$$

For the example array considered above (where $N = 11$, $d = 0.1$) we calculate the first minor lobe to be at $f = 514.5$ Hz, for which $kd = 0.942$, which is in agreement with figure 16.14.

16.2.2 Through the $kd\sin\theta$ Lens


Figure 16.19: Distortion due to stretching.

So far we have been looking at our directivity as a function of this new variable Ω. But remember, this is not an angle. It is related to the angle through the $\sin$ function, i.e. there is a non-linear mapping between the true angle θ and our angular variable Ω.

Recall that for small values, the sin function is quite linear. And so we might expect that around 0 degrees, the true angular response is quite similar. However, around the 90 degree region the sin function really stretches the values out.

What does this mean for our directivity? Well, turns out that the secondary lobes are much wider than the main lobe! See figure 16.20.

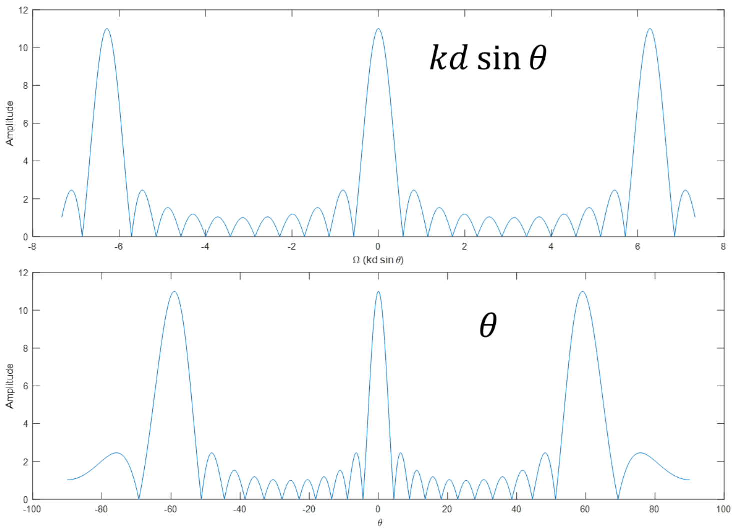


Figure 16.20: Line array directivity in terms of angle θ and angular variable Ω.

17 Fourier Tricks

Now that we have introduced our mathematical model for a line array in the far field, and discussed its interpretation as the discrete Fourier transform of an array's amplitude coefficients, we can look at some of the benefits that come with this interpretation.

17.1 Beam Steering

Right, lets think back to our friend Hugen, and his useful principle of wave super position. He said that we could recreate any wave front by populating it with a sufficient number of monopoles. In theory we would not be able to tell the difference between the original and recreated waveform.

Now suppose we want to steer the sound in a particular direction. How might we do this? Well we could just rotate the array. But think about it, by rotating the array we are simply adjusting the path lengths over the array, so that some are more delayed than others. We can use this idea to steer the array directivity

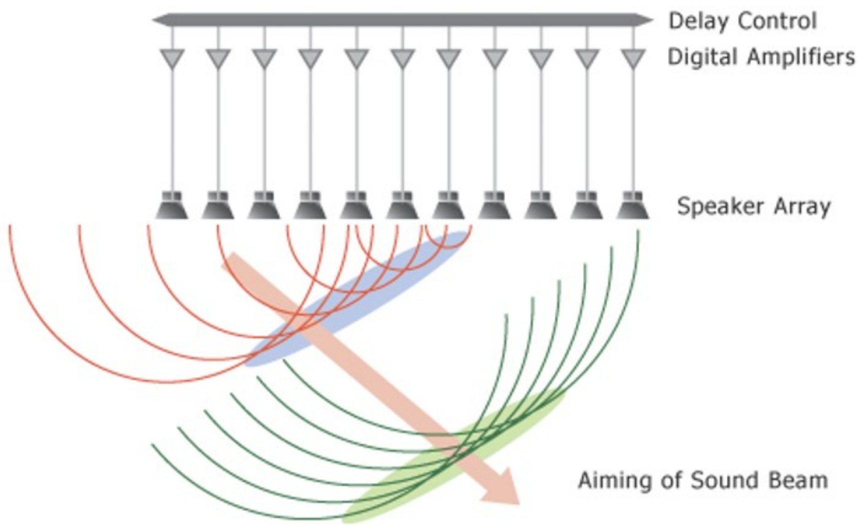


Figure 17.1: Line array beam steering.

electronically! This is what we call beam steering. How do we do it? Simple, calculate the additional path length for each speaker that is introduced by a rotation of a given angle. Now apply an electronic phase delay to each speaker that corresponds to this additional path length! This will steer the direction of the radiated wave front.

This is a very useful idea, particularly to fine tune beam direction, for example to compensate for environmental factors, e.g. wind changes the speed of sound a bit.

So, based on our Fourier description array directivity, how might we think about implement some beam steering? **The shifting theorem!**

Remember, the shift theorem tells us that delaying a signal in the time domain by t_0 is the same as multiplication with a complex exponential $e^{j\omega t_0}$ in the frequency domain. This is the same as adding a linearly increasing phase lag with frequency.

Now, if we want to shift something in the spatial domain (i.e. rotate beam) we can use the same idea. We simply multiply the complex amplitudes by a progressive (linear) phase change. Mathematically,

$$p_T(\Omega) = \sum_{n=0}^{N-1} A_n e^{jn\Omega(\theta)} e^{jn\Delta\Omega} = \sum_{n=0}^{N-1} A_n e^{jn\Omega(\theta)} e^{jnkd \sin\Delta\theta} = \sum_{n=0}^{N-1} A_n e^{jn\Omega(\theta)} e^{jn\omega\tau} \tag{17.1}$$

where $\tau = \frac{d}{c}\sin\theta$ is the necessary time delay between neighbouring monopoles to achieve a rotation of angle θ.

By doing this the main lobe moves off axis, however the visible region stays the same. So because of the non-linear scaling of the Ω term, the beam becomes more smeared at greater angles.

Shown in figure 17.2 are the directivities of the previous line array example for 3 different cases; no beam steering, $30°$, and $60°$.

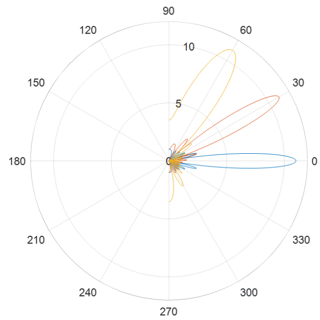


Figure 17.2: Line array beam steering.

Beam steering is a really useful feature of line arrays. Often it is advantageous to direct the radiated sound in a particular direction, e.g. towards and audience. In outdoor spaces this would reduce the propagation to nearby dwellings, thus avoiding complaints! In large indoor spaces it can be used to increase the direct to diffuse ratio (i.e. less of the radiated sound ends up reverberation) thus improve clarity and intelligibility. What about in the home? Well there are some interesting applications using sound bar technology.

Figure 17.3: Sound bar loudspeaker.

Using an array of drivers (see for example figure 17.3) is possible to create a beam of sound, and steer it such that it is reflected of a surface before arriving at the listener position. To the listener, it will sound as if a virtual speaker were located at the position of reflection. This idea can be used to create virtual surround sound systems, ass illustrated in figure 17.4. But how do we create multiple speakers if we only have one beam?

Figure 17.4: Virtual 5.1 system using beam forming.

Remember, the Fourier transform satisfies the principle of super position. This means that we can super impose multiple beams on top of each other. Even better, we can individually steer each of these beams.

17.2 Source Directivity

So far we have been considering a line array made up of a finite number of simple monopole sources. Whilst this is a reasonable assumption at low frequencies, in the mid frequency range its not quite true. We have also seen that as you go up in frequency, our loudspeaker array gets more and more directional, i.e. the visible region gets wider. Eventually we run into aliasing problems and major side-lobes appear. How can we combat this aliasing issue?

We have already seen that the visible region is dictated by the value of kd. So if we want to make the visible region smaller (i.e. get a broader directionality) we need to space the sources closer together (make d smaller). The problem is that in PA systems you need big loudspeakers!

How else can we avoid the issue of aliasing? One way is to use a directional loudspeaker for each of the array elements, i.e. replace each monopole source with a source whose directivity is more narrow. To do this we need to acknowledge that the replacement of a monopole with a more directional source is a form of spatial filtering, which is equivalent to a spatial convolution of the two sources. This is quite tricky to visualise, luckily however, we can call on another property of the Fourier transform, **the convolution theorem**; i.e. convolution in the spatial domain is multiplication in polar domain.

So to replace the monopole elements of our line array with more directional sources, we simply need to multiple the polar response of line array, with the polar response of the new source element. Lets look at an example.

Suppose we want to replace each of the monopole elements in our previous array with a short line source, as in figure 17.5. If the length of each line source is set as the monopole spacing d, the line array should yield the same directivity as a continuous line source of length L.

The radiated pressure from each line segment if obtained by integrating a monopole source over line of the appropriate length,

$$p_{\Delta L} = \int_{-d/2}^{d/2} \frac{A}{r} e^{j(\omega t - kr)} dr \rightarrow p_{\Delta L} = \frac{A}{r} \frac{\sin\left(\frac{1}{2}kd\sin\theta\right)}{\frac{1}{2}kd\sin\theta}. \tag{17.2}$$

Similarly, the radiated pressure from the continuous line source of length L is given by,

$$p_{L} = \int_{-L/2}^{L/2} \frac{A}{r} e^{j(\omega t - kr)} dr \rightarrow p_{L} = \frac{A}{r} \frac{\sin\left(\frac{1}{2}kL\sin\theta\right)}{\frac{1}{2}kL\sin\theta}. \tag{17.3}$$

Shown in figure 17.6 are the polar responses of the monopole line array and the line segment source. Notice that for a line source of length d the nulls coincide exactly with the side lobes of the monopole array. Now, by swapping out the omni-directional source for a directional line source we are essentially applying a spatial filter. It is like convolving the spatial response of the short line source over the monopole array. Admittedly, this is quite difficult to visualise. But remember, convolution in the spatial domain is the same as multiplication in the polar domain. So to get the polar response of our line segment array we just

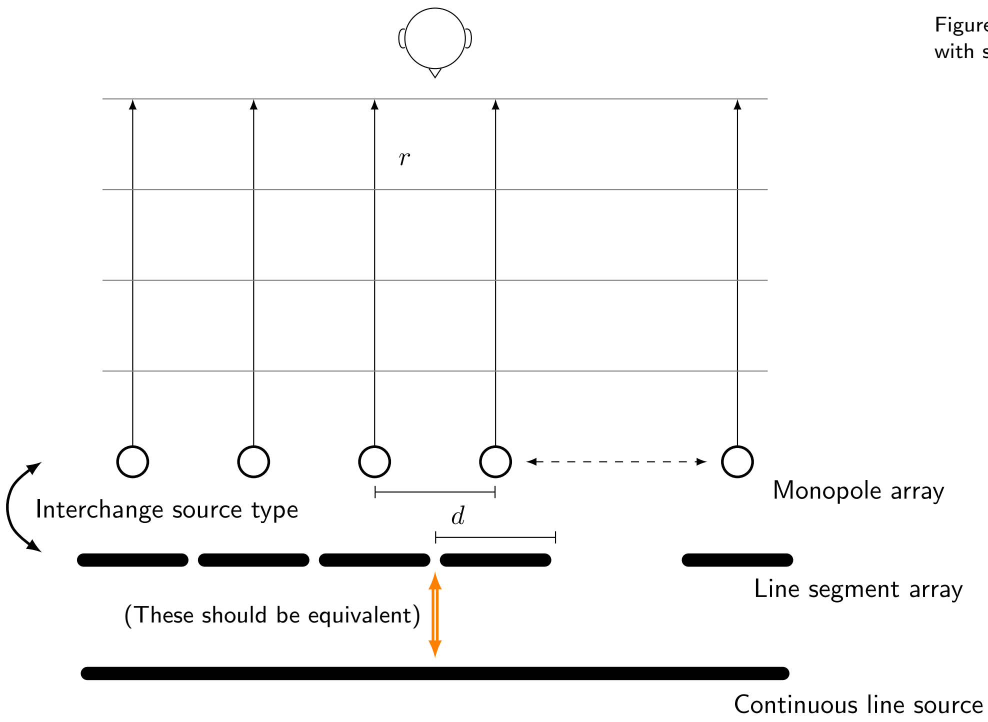


Figure 17.5: Replacement of monopole sources with short line segment sources.

multiply to two polar responses together. What we get is shown in figure 17.7. Also shown (orange dashed line) is what we would get from a single continuous line source of length L. Notice that they are exactly the same, as expected.

Also, notice how by using a line segment source we have completely removed the side lobes (this is equivalent to the filtering process applied in AD conversion!). Clearly using continuous line source is not possible in reality, so to control directivity we generally use horn loaded loudspeakers.

17.3 Array Shading

We have see that as you go up in frequency the polar response of our line array gets more narrow, as the visible region gets wider. We have also seen that as the array gets longer the array also gets more directional (see for example figure 17.8); the first anti-lobe occurs when the radiated wave length is equal to the array length, so the longer the array the earlier you run into a anti-lobe. More obviously, an array of length 1 (i.e. a single monopole) is omni-directional. By adding additional sources the directivity can only get more directional.

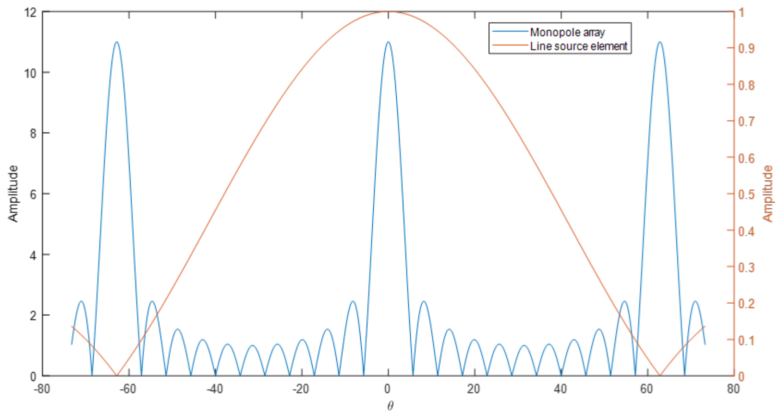


Figure 17.6: Polar response of monopole line array and line segment source.

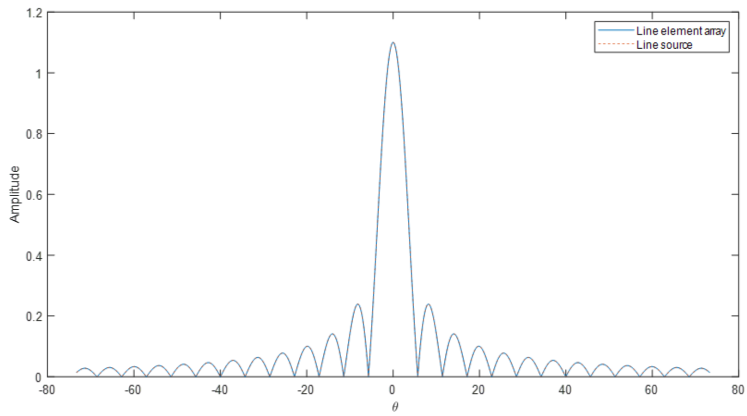


Figure 17.7: Polar response of line segment line array and continuous line source.

In an ideal world what we want is equal directivity at all frequencies. How can we achieve this? At high frequencies we have to rely on horns to control the loudspeaker's directivities, and so the beam width. But at moderate frequencies we can use a technique call array shading.

Array shading is quite a simple idea. At low frequencies we use all of the loudspeakers in the array. At high frequencies, to compensate for the narrower beam width, we use less loudspeakers in the array. This has the effect of broadening the beam width. Lets look at an example.

Shown in figure 17.9 is the polar response of a 21 element line array, modelled at 250 Hz and 1kHz. Notice that the 1 kHz response is much more narrow than at 250 Hz. In figure 17.10 we have the polar response of the same array, but for the 1 kHz response we only include the 5 central array elements. Notice that the beam width is now approx. the same as at 250 Hz. This is what we call array shading.

In reality we may want to apply some smooth windowing function (rather than just on/off) across the array amplitudes to control the beam width more carefully, e.g. a Hanning window as in figure 17.14.

So what does this look like mathematically? Well, the application of array shading amounts to applying a window to the amplitudes of a line array. This

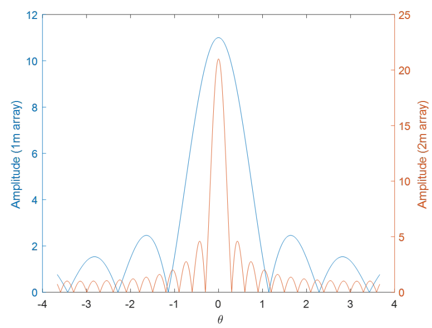


Figure 17.8: Example of beam narrowing due to increase array length. 1 meter array vs. 2 meter array.

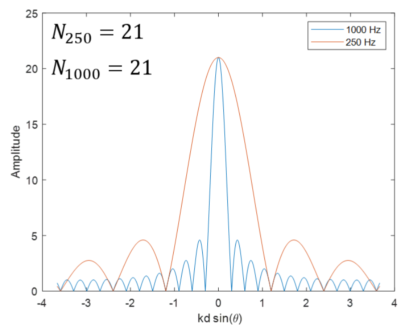
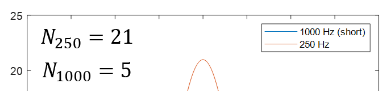


Figure 17.9: 21 element line array, modelled at 250 Hz and 1kHz



is equivalent to multiplying the spatial monopole array, with a spatial window function. Now according to the Fourier transform what is this equivalent to? A convolution in the polar domain!

So multiplication of the array amplitudes by a spatial window, is equivalent to convolving their respective polar domain responses. To do this we first need to figure our what their individual polar domain responses are. Shown in figure 17.11 is an example.

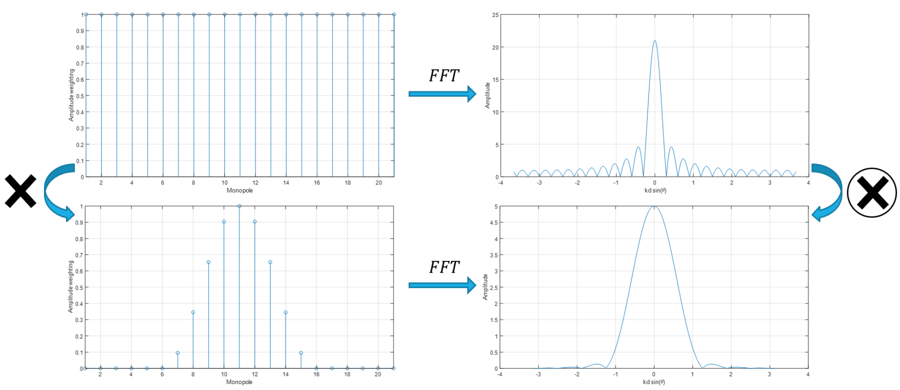


Figure 17.11: Example array shading as a convolution in the polar domain.

In the top left of figure 17.11 we have our non windowed array of amplitude coefficients. In the top right we have its corresponding polar domain response. Looks like a sinc function right? Now bottom left we have our spatial window, and on the bottom right we have its polar domain response. Looks a little like a sinc function, but not really.

Now according to the Fourier transform properties, multiplication of the left hand plots is equivalent to convolving the right hand plots. Shown in figure 17.12 is what happens if we actually apply this window. On the left is the original array response, and on the right is the response with shading applied. Notice we get a beam width approx. equal to the window polar response, which is much wider than the original array response.

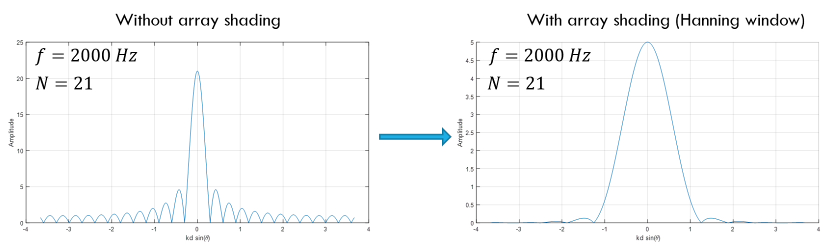


Figure 17.12: Result of array shading.

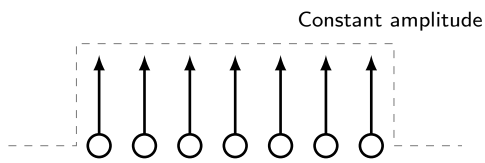


Figure 17.13: Line array with no array shading.

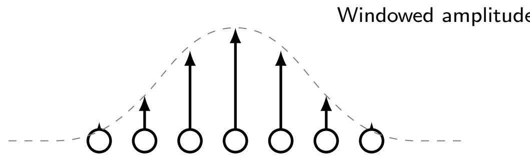


Figure 17.14: Line array with smooth Hanning window array shading.

Admittedly, the level of the output has been reduced somewhat, but this can be compensated for using standard filtering techniques.

17.3.1 Optimisation

This array shading idea is all well and good, but we just end up adjusting the beam width at all frequencies, low frequencies included. We want to be able to apply a different shading at different frequencies.

One solution is to implement some complicated DSP. This is exactly what the Intellivox array by duran audio (now JBL) does. The idea is that every driver has its own signal path, including controllable delay, filtering and amplification. This means that shading can be made to be frequency dependant! I.e. you can control the apparent length of the array for each frequency.

So for example you may end up with filters that look flat in the centre of the array and are low pass at the edges. This would mean that low frequencies you use the whole of the array, whilst at high frequencies you would use mostly the centre of the array. The tricky part is figuring out the right filters to use.

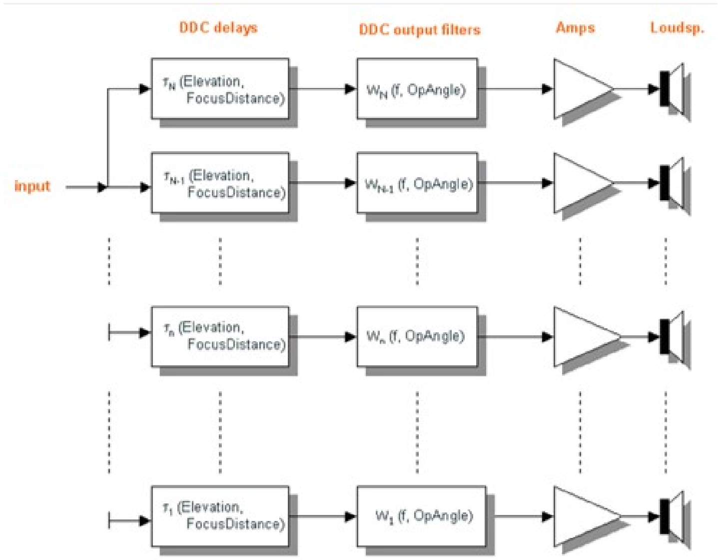


Figure 17.15: The Intellivox array by duran audio (now JBL).

These will change for every space! This is a great example of where numerical optimisation can help. The basic idea is that you define some figure or merit (or a cost function). This could be the beam width, or perhaps the ratio of main to side lobe level, or even STI over the room. Then you adjust the model parameters (filter response / delays) such that we minimise (or maximise) the figure of merit.

Numerical optimisation is a very very broad topic, and a great many optimisation algorithms exist. Some notable ones are: 'Genetic Algorithm', 'Simulated Annealing', 'Simplex' and 'Stochastic hill climbing'.

17.3.1.1 Genetic Algorithm Example

I have put together a little demo of how you might try to implement an optimisation algorithm for array updating. You can find the MATLAB script on blackboard. It uses the Genetic Algorithm. So lets quickly talk about how this algorithm works.

The genetic algorithm is designed to mimic the way in which positive traits are based through generations, i.e. natural selection. To start of the process we have to generate a random population of parameters (e.g. a number of sets of random filter coefficients). At each step in the optimisation a number of individuals (e.g. one set of filter coefficients) are selected at random from the current population. These are so called parents. The parents are used to produce the children for the next generation. Over successive generations, the population "evolves" toward an optimal solution.

The genetic algorithm uses three main types of rules at each step to create the next generation from the current population:

1) Selection rules select the individuals, called parents, that contribute to the population at the next generation. E.g. select a proportion with lowest cost function (best directivity?)

2) Crossover rules combine two parents to form children for the next generation. This could be done by averaging the individual parameter values, or perhaps splicing them.

3) Mutation rules apply random changes to individual parents to form children. They add some random factor to encourage mutation. Beneficial mutations are then adopted by the population.

After a certain amount of elapsed time, or generations, the algorithm outputs the optimal solution. In the context of array optimisation, the following steps are followed:

1) Simulate array using filters properties determined by a random set of coefficients.

2) The radiated pressure from the array is calculated in the far field.

3) A figure of merit characterizing the directivity is calculated from the pressure distribution.

4) The filter properties are altered by changing the coefficients according to a standard algorithm that searches for a minimum in a variable.

5) Steps (2) to (4) are repeated until minimum in the figure of merit is found.

I urge you to have a play with the example on blackboard.

18 Low Frequency Directivity

Now that we have dealt with mid/high frequency directivity, lets have a think about low frequency directivity. It turns out we will have to tackle this in a completely different way. Lets start by looking a dipole configuration of two loudspeakers.

18.1 Dipole Radiation

At low frequencies we can model a dipole loudspeaker arrangement as two monopoles radiating out of phase, spaced some distance d apart (see figure 7.3). Physically we can get a similar configuration by taking a loudspeaker and attaching to its rear end a duct of length d. Remember, the front and rear of a loudspeaker driver are always out of phase!

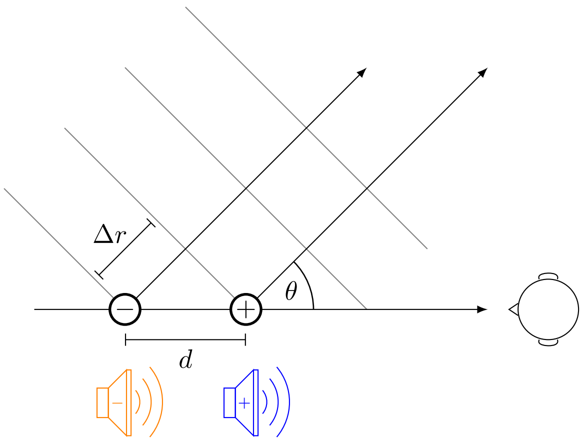


Figure 18.1: Dipole loudspeaker arrangement for low frequency directivity.

Now lets think about far field radiation of this loudspeaker configuration. Like with out line array, by considering the far field response we can assume plane wave propagation. This time however, we are going to measure θ slightly differently. Why? Because the front and rear of our dipole loudspeaker are in line with the listener position. So really the problem is very similar to what we did previously, just with a different angle (and just two monopoles).

Now lets derive the radiation from this arrangement, considering low frequency ($kd << 1$) first.

18.1.1 Low Frequency

From the geometrical set up of our problem the additional path length travelled by the rear radiated sound is given by,

$$\cos\theta = \frac{\Delta r}{d} \to \Delta r = d\cos\theta. \tag{18.1}$$

The radiated pressure due to the front monopole is given by,

$$p_f = A_0 e^{j(\omega t - kr)} \tag{18.2}$$

where r is the distance to the listener position. The rear radiated pressure is similarly given by,

$$p_b = -A_0 e^{j(\omega t - k(r+\Delta r))} = -A_0 e^{j(\omega t - k(r + d\cos\theta))}. \tag{18.3}$$

The rear pressure can be expressed in term of the front radiated pressure by separating the complex exponential,

$$p_b = -A_0 e^{j(\omega t - kr)} e^{-jkd\cos\theta} = -p_f e^{-jkd\cos\theta}. \tag{18.4}$$

The total radiated pressure is then given by,

$$p_T = p_f + p_b = p_f \left(1 - e^{-jkd\cos\theta}\right). \tag{18.5}$$

Now lets use Euler's formula to express the complex exponential,

$$p_T = p_f \left(1 - \left[\cos\left(kd\cos\theta\right) - j\sin\left(kd\cos\theta\right)\right]\right). \tag{18.6}$$

Now since we are considering low frequencies we can assume that $kd << 1$. Taking first order approximations for the trig terms ($\sin x \approx x$ and $\cos x \approx 1$) leads to,

$$p_T = p_f \left(1 - \left[1 - jkd\cos\theta\right]\right) \tag{18.7}$$

which simplifies to,

$$p_T = p_f jkd\cos\theta. \tag{18.8}$$

Equation 18.8 represents the total radiated sound pressure level from our dipole source. It is made up of two terms. First we have the total radiated pressure of a monopole, p_f, multiplied by a frequency dependent term jkd. Second we have an angular dependence in the form of $\cos\theta$.

So what's interesting about this? By introducing a second (out of phase) monopole we have introduce a linear dependence on frequency! Now if we were to double the frequency, we would get double the radiated pressure. This corresponds to a $+6$ dB per octave increase. This is kind of bad news for the low frequencies... When f is very small, we get very little radiated pressure. Now very good for a sub woofer! That said, we can always EQ out this frequency dependence, by applying a low frequency boost at -6 dB per octave.

So looking at the radiated pressure we have two important terms. One describes the efficiency of radiation (frequency dependant part), and the other describes the directivity of radiation (angle dependent part). These are shown in figure 18.2. The directivity is determined by the cos term. This gives us what is called a figure of 8 polar pattern, i.e. large response on axis, but nothing at $\pm 90°$.

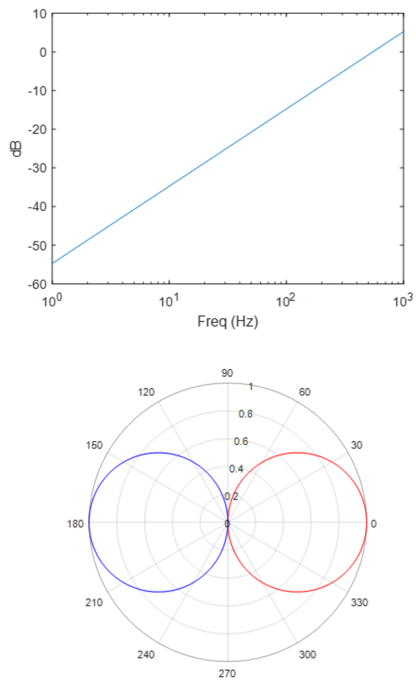


Figure 18.2: Frequency and angular dependence of total pressure/pressure gradient.

The main thing to acknowledge is that with a dipole source, there is a lot of cancellation going on. Most of it is at $\pm 90°$. Towards the front and rear the spacing between the two sources introduces an extra phase which lets some sound 'escape'. But there is still quite a bit of cancellation at low frequencies – hence the need for some corrective EQ!

18.1.1.1 Practical Dipoles

Another way of thinking of a dipole is by taking a loudspeaker and placing it in some sort of enclosure, e.g. a duct or finite baffle.

For the ducted loudspeaker this has the effect of isolating the direct radiation from each side. The length of the duct would be d. The radiation at the end of the duct would appear as if it were a second source. This would lead to the same radiated response as before, i.e. a figure of 8.

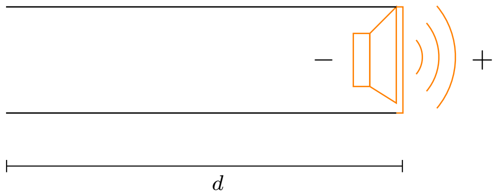


Figure 18.3: Dipole source as loudspeaker in a duct.

There is another way of achieving this. If we put a loudspeaker it in some finite size baffle at low frequencies the radiated sound will happily diffracts around the baffle. In doing so it will travel some additional path length. The exact length of this path is not as clear, but the same effect happens. We have a phase difference between two out of phase sources. This gives us a figure of 8 pattern.

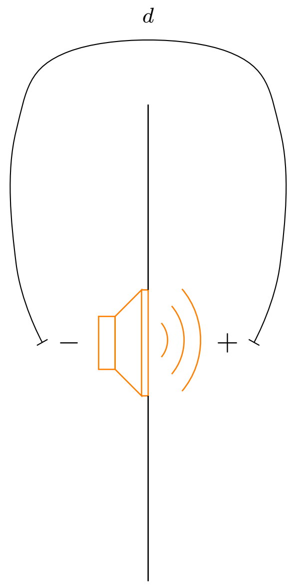


Figure 18.4: Dipole source as loudspeaker in a baffle.

So if we put a large driver in a small baffle we don't get a lot of bass. So it is looking pretty rubbish for a sub-woofer. But as mentioned before, it is possible to equalise this problem out. Another problem is that a figure of 8 directivity isn't actually very useful. Why would we want to radiate sound out the back of a loudspeaker, when the audience is in front? Seems quite odd thing to do at the moment, but we are moving towards something.

18.1.2 High Frequency

Although we are interested primarily in low frequencies right now, it is worth having a quick look at how high frequencies behave, especially as this whole dipole concept applies equally to microphones as well (we will get to that later on in the notes).

To make life a little simpler, lets consider the on axis response (so in other words when $\theta = 0$) and go back to equation 18.6, before we made the low frequency approximation,

$$p_T = p_f \left(1 - \left[\cos\left(kd\cos\theta\right) - j\sin\left(kd\cos\theta\right)\right]\right). \tag{18.9}$$

Now lets look at how this changes with frequency.

We are only interested in the frequency dependant (directivity) part, so lets take the magnitude of this term.

$$|DF| = |1 - [\cos kd - j\sin kd]| = \sqrt{(1-\cos kd)^2 + \sin^2 kd} \tag{18.10}$$

Now a little bit of algebra. First expanding the squared bracket,

$$|DF| = \sqrt{1 - 2\cos kd + \cos^2 kd + \sin^2 kd} \tag{18.11}$$

the using the identity $\cos^2 + \sin^2 = 1$,

$$|DF| = \sqrt{1 - 2\cos kd + 1} = \sqrt{2 - 2\cos kd} = \sqrt{2(1-\cos kd)}. \tag{18.12}$$

Finally we arrive at a frequency dependent directivity factor (although we are assuming on axis),

$$|DF| = \sqrt{2}\sqrt{1 - \cos kd}. \tag{18.13}$$

Recall that $k = 2\pi/\lambda$. Substituting this into the above we can see that this directivity factor will tend to zero whenever the wave length is an integer multiple of the spacing,

$$|DF| = \sqrt{2}\sqrt{1 - \cos\frac{2\pi d}{\lambda}}. \tag{18.14}$$

This makes sense right? By the time the rear radiated sound has reached the front source, it is $180°$ out of phase, and so cancels completely.

We can also see that when the ratio is an integer wave length plus a $1/2$, we get a maximum. So we have that for,

$$\frac{d}{\lambda} = 1, 2, \cdots \tag{18.15}$$

and

$$\frac{d}{\lambda} = \frac{1}{2}, \frac{3}{2}, \cdots, n + \frac{1}{2} \rightarrow |DF| = 2. \tag{18.16}$$

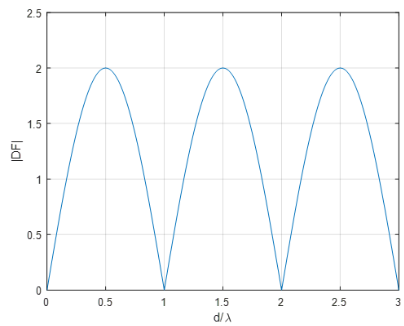


Figure 18.5: On axis dipole response as a function of $d/\lambda = fd/c$.

Plotting this response for all d/λ we get something that looks a lot like a comb filter.

So have seen that at high frequencies (on axis) we get a comb filtered response from our dipole source. This arises due to the constructive and destructive interference from the two sources.

When the frequency is such that the wavelength is equal to the spacing we get a $360°$ phase difference, but because sources are out of phase with each another to start with, the phase difference at listener position is only $180°$, and so we get complete cancellation. This clearly repeats at integer multiples of this wavelength.

When the freq is such that the $1/2$ wavelength is equal to the spacing you get a path difference that accounts for the $180°$ phase shift, plus the extra $180°$ phase shift due to the two sources being driven out of phase. This means at the listening position we a phase shift of $360°$, so everything is back in phase and we get constructive interference. This also repeats at integer multiples of this $1/2$ wavelength.

When designing a low frequency loudspeaker we clearly want some sound. So we are interested in up to perhaps the first peak, i.e. when spacing is equal to $1/2$ wavelength. In fact for subwoofers you will usually design them so that the spacing is equal to a $1/4$ wave length of the centre of the operating band.

18.2 Controllable Directivity

The main disadvantage of a dipole source is due to the low frequency roll off arising from the jkd term in its response. Also, remember that the piston impedance function has a $(ka)^2$ term in its real part. This means we get lower radiated power as frequency decreases. The only way to compensate for this is to increase the surface area, i.e. to overcome impedance function you need a very large diaphragm. Coupled to that, you need a $+6$ dB per octave bass boost to overcome the linear frequency dependence introduced by the dipole source. So it's a bit of a trade off. By introducing a second out of phase source, you get an improved directionality but as a consequence the radiated power and loudspeaker efficiency is reduced.

Interestingly, for a microphone it is possible to compensate for this $1/f$ roll off by adjusting the mechanical parameters i.e. play with the resonance of the diaphragm to tune the system. This is clearly preferred to electrical compensations (we just want to plug in and get flat response).

We have seen that a dipole loudspeaker configuration will give us a figure of 8 directivity pattern. Although not particular useful by itself, this directivity is incredibly useful when combined with an omni-directional response as well. By controlling their relative contributions, we are able to control the directional response of the loudspeaker.

18.2.1 Cardioid Response

We have looked at the dipole loudspeaker and seen its figure of 8 directivity. This is interesting, but there are far more useful directivities out there than the figure of 8. In fact, it turns out that by combining the figure of 8, with an omni directional response, we can create a whole load of useful directivities. But first lets do a bit of recapping. We've got the omni-directional pattern, where a loudspeaker radiates at all angles equally. We have a figure of 8 pattern, where we only get radiation forwards and backwards.

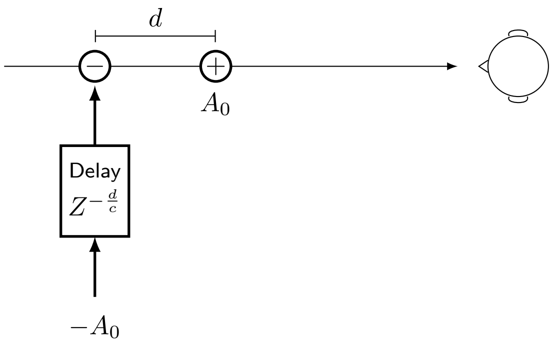


Figure 18.6: Dipole configuration with a delayed rear source.

Another really useful directivity is the so called cardioid pattern. This is where the sound is mostly directed in one direction. It turns out that we can design a directivity like this by introducing a delay into the system. In particular for a cardioid pattern, we set this delay to be equal to the distance between the sources.

Recall the radiated pressure from a dipole source,

$$p_T = p_f + p_b = p_f \left(1 - e^{-jkd\cos\theta}\right). \tag{18.17}$$

Introducing a delay of length d is equivalent to multiplying by an additional complex exponential as so,

$$p_T = p_f \left(1 - e^{-jkd\cos\theta} e^{-jkd}\right). \tag{18.18}$$

Using Euler's formula we get,

$$p_T = p_f \left(1 - [\cos(kd\cos\theta) - j\sin(kd\cos\theta)][\cos(kd) - j\sin(kd)]\right). \tag{18.19}$$

Now again we consider far field radiation by assuming $kd << 1$ and taking first order approximations for the trig terms,

$$p_T = p_f \left(1 - [1 - jkd\cos\theta][1 - jkd]\right), \tag{18.20}$$

whilst ignoring any terms that are second order in kd,

$$p_T = p_f\left(1 - [1 - jkd\cos\theta - jkd]\right). \tag{18.21}$$

Finally we arrive at,

$$p_T = p_f\left(jkd\cos\theta + jkd\right) = p_f jkd\left(1 + \cos\theta\right), \tag{18.22}$$

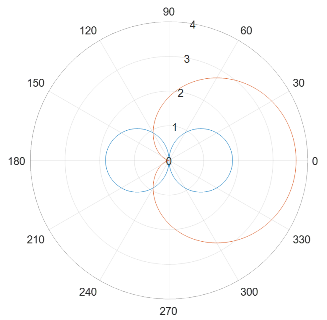


Figure 18.7: Cardioid response

an expression similar to what we had before, except there is an extra factor of $+1$. This extra $+1$ term can be though of as introducing an additional monopole behaviour (albeit one that is frequency dependent). The $\cos\theta$ then corresponds to a dipole.

So what does this look like? Shown in figure 18.7 is the resultant response. By adding a delay that is equivalent to the path length difference between the two sources get a cardioid (heart shaped) response. So why does this happen?

Let's consider the backwards radiation first. Our starting point is a dipole, where the rear source is set 180° out of phase with the front source. The difference in path length between the two (in the backwards direction) introduces a phase delay e^{jkd}. The additional delay applied to the rear source, e^{-jkd}, effectively reverses the path length phase delay. As a result it is as if the two sources are located in the same position. Hence, they interfere destructively and we get no radiation backwards.

What about the front radiation? The difference in path length between the front and rear sources (in the forwards direction) introduces a phase delay e^{-jkd}. Unlike the rear radiation (where the applied delay *reverses* the path length delay), in the forward direction the applied delay *combines* with the path length delay. Together these avoid the 180° phase difference, leaving behind some frontward radiation. For example, suppose we have a spacing d equal to a $1/4$ wavelength.

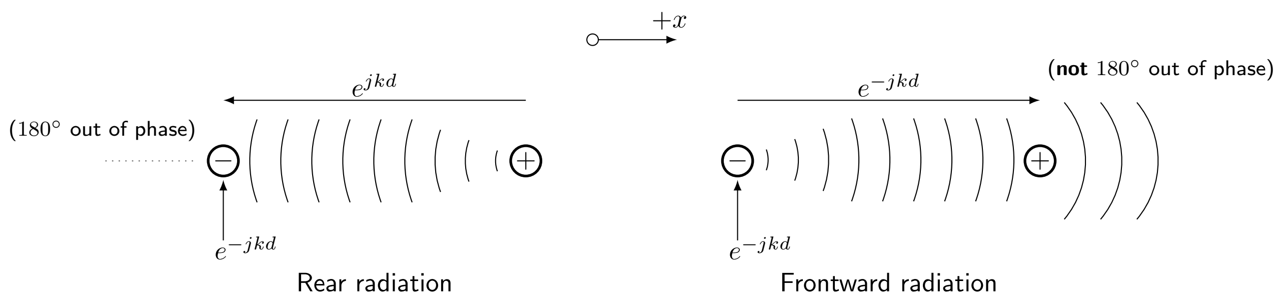


Figure 18.8: Front and rear response of cardioid directivity.

Backwards, the rear source is 90° ahead of the front source radiation due to the path length difference. This cancels the -90° phase delay applied to the rear source. All that's left is the -180° delay due to the inversion of the rear source. The front and rear sources are out of phase; they cancel completely.

Forwards, the front source is 90° ahead of the rear source radiation due to the path length difference (equivalently, the rear source is -90° behind the front source). This combines with the -90° phase delay applied to the rear source to give a -180° phase delay. When combined with the 180° delay due to the inversion of the rear source, the front and rear sources end up completely in phase; they interfere constructively giving a large boost in the frontward radiation.

18.2.2 Variable Directivity

We have seen that the cardioid pattern is made up of two parts. One part omni, one part figure of 8. We got this pattern by considering a rear source delay that was equal to the source spacing d. Now lets consider what happens if we change this delay.

Recalling equation 18.22, and introducing d_Z to denote the additional delay, the response of our variable delay dipole source is given by,

$$P_T = P_f \left(jkd \cos\theta + jkd_Z \right). \tag{18.23}$$

Clearly by setting the delay to $d_Z = 0$ we arrive at the figure of 8 response of a dipole. By setting $d_Z = d$ we arrive back at the cardioid response. You could push d_Z a little further to obtain a sub-cardioid response, but remember we derived equation 18.23 by assuming $kd << 1$. this assumption applied also to the delay d_Z. So we cant use this equation for large values of the delay.

In between the figure of 8 and cardioid response we have a whole load of different directivity patterns, including the super-cardioid and hyper-cardioid responses (see for example figure 18.9), all obtainable by simply altering the rear source delay.

So by controlling the delay applied to the rear loudspeaker (when wired antiphase), we can alter the contribution of the omni and figure of 8 response terms, and in turn control the low frequency directivity of our source.

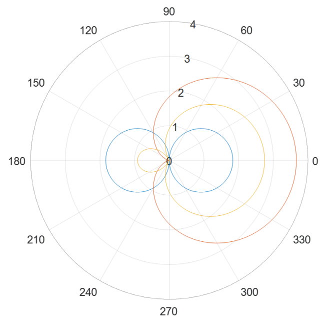


Figure 18.9: Polar response of dipole source with variable delay. Figure of 8, super cardioid, and cardioid polar patterns.

18.2.3 Directivity vs. Frequency

In deriving equation 18.23 we assumed $kd << 1$. Lets now relax that assumption and look a little more closely at the issue of directivity vs. frequency. Taking equation 18.19 (i.e. before we assumed $kd << 1$),

$$p_T = p_f \left(1 - [\cos(kd\cos\theta) - j\sin(kd\cos\theta)][\cos(kd) - j\sin(kd)]\right). \tag{18.24}$$

but considering only the on axis response ($\theta = 0$) we get,

$$p_T = p_f \left(1 - [\cos(kd) - j\sin(kd)][\cos(kd) - j\sin(kd)]\right), \tag{18.25}$$

or by substituting $k = 2\pi/\lambda$,

$$p_T = p_f \left(1 - \left[\cos\left(2\pi\frac{d}{\lambda}\right) - j\sin\left(2\pi\frac{d}{\lambda}\right)\right]^2\right). \tag{18.26}$$

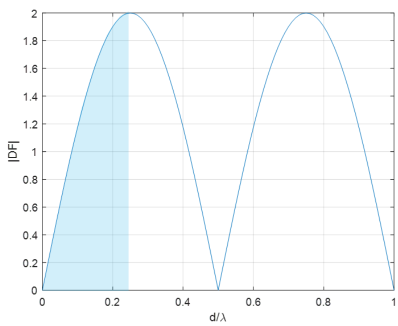


Figure 18.10: On axis frequency response of dipole source with decreasing wave length.

We can now have a look at what happens to this equation as we go up in frequency, or as the wave length λ decreases.

We are interested in the maximum and minimum values of equation 18.26. Starting with the minimum it is clear that this is achieved when the squared bracket term is equal to 1. Since the bracket is squared, this will occur when the term inside is either ± 1. When the argument of cos is an integer multiple of π the cos term is 1. For this same argument the sin term is 0. So, when $\lambda = 2d$ the cos argument equals π. The cos term then equals 1. For an argument of π the sin term equals 0. The total pressure is therefore 0.

$$\lambda = 2d \rightarrow 2\pi\frac{d}{\lambda} = \pi \rightarrow \cos\pi = -1 \rightarrow P_T = 0 \tag{18.27}$$

$$\lambda = 2d \rightarrow 2\pi\frac{d}{\lambda} = \pi \rightarrow j\sin\pi = 0 \rightarrow P_T = 0 \tag{18.28}$$

For a maximum we need the $\sin$ term to equal 1. This will give us a $-j^2 = -1$ term which will then yield $+2$ overall. When the $\cos$ and $\sin$ arguments are $\pi/2$ the $\cos$ term will be 0, and the $\sin$ term will be 1. This is achieved when $\lambda = 4d$. At this frequency the radiated pressure is twice that of the single driver alone.

$$\lambda = 4d \rightarrow 2\pi\frac{d}{\lambda} = \frac{\pi}{2} \rightarrow \cos\frac{\pi}{2} = 0 \rightarrow P_T = 2 \tag{18.29}$$

$$\lambda = 4d \rightarrow 2\pi\frac{d}{\lambda} = \frac{\pi}{2} \rightarrow j\sin\frac{\pi}{2} = j \rightarrow P_T = 2 \tag{18.30}$$

As the wave length continues to decrease (with increasing frequency) integer multiples of the above occur, and we get a repeating comb like pattern in the frequency response. How this decreasing wave elngth effects the directivity is shown in figure 18.11. We cycle between cardioid and figure of 8 directivity. Notice that the maximum output $P_T = 2$ first occurs when $\lambda = 4d$, or when $d = \lambda/4$. According to figure 18.11 this corresponds to a cardioid directivity. Similarly, the minimum output $P_T = 2$ first occurs when $\lambda = 2d$, or when $d = \lambda/2$. According to figure 18.11 this corresponds to a figure of 8 directivity, but facing in the off-axis direction.

The region highlighted in figure 18.10 corresponds to the region in which the spacing d is less than $1/4$ a wavelength. It is only this region that is useful. If we space the sources too closely together this $1/4$ wave length frequency will be very high. Although that sounds good, as it extends the useful region, it actually means we get a very small output at low frequencies (the phase delay between the two drives will be very small, and so almost perfectly out of phase)! This would require a huge boost at low frequencies to make it work.

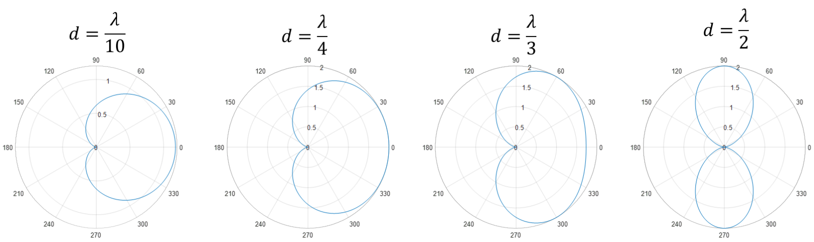


Figure 18.11: Polar response of dipole source with decreasing wave length.

Instead what we want is a large spacing, so that we can get a large output at low frequencies. Sub-woofers are usually designed such that the is $1/4$ wavelength at the centre of operating region. For example, if we wanted a 50 Hz sub the wavelength is 6.8m, and so $1/4$ wavelength is 1.7 m! (that's big for a loudspeaker, but perfectly reasonable in a PA system)

18.2.4 What About Microphones?

Things are looking pretty good for our low frequency directivity. It turns out that we can build a sub-woofer array that doesn't have to be that big. At 50 Hz, $1/4$ wavelength is 1.7 m. If you are using two speakers this is the distance apart they would need to be. This is perfectly reasonable in a PA system. With a HiFi system on the other hand, clearly this spacing needs to be lower.

So here is the general idea. Feed a spaced pair of (out of phase) sub-woofers with same signal, one of which is delayed (see figure 18.12). Given the spacing d, it is easy to work out what the extra delay should be to achieve a cardioid response. Then you can just reduce the delay to get a more direction front beam, with the trade off being that you get some rear radiation also.

It turns out that all of this directivity stuff applies equally as well to microphones. With a microphone what you can do is place the diaphragm at one end of a tube. This way the total response at the diaphragm is the superposition of the direct sound and a time delayed version (which is $180°$ out of phase since it is applied to the rear of the diaphragm). The path length through the tube is equivalent to our rear source delay. It either cancels or combines with path length around the outside of the tube to cause destructive or constructive interference on the diaphragm. And just like our controllable source directivity, if we vary the diaphragm position we alter the corresponding delay, and so change the microphone directivity.

Another microphone based approach would be to use two separate diaphragms, delaying the signal from one before adding them together, just like our sub-woofer directivity. This would also create a controllable microphone directivity pattern. In fact, with a microphone we have a bit more flexibility in the design, in particular we have more control over where the resonance of the diaphragm is. With a loudspeaker we are pretty much stuck with it being very low (because the driver is quite massive). We will cover microphone directivity in more detail in Part IV of the notes.

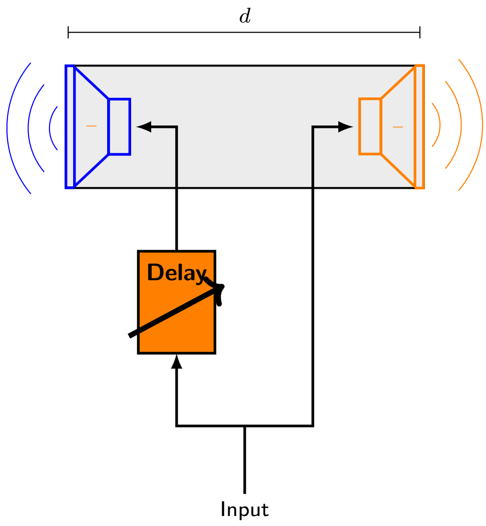


Figure 18.12: Variable delay sub-woofer.

Part IV

Microphone Design

19 Introduction

We have already discussed the idea that microphones and loudspeakers are very similar. Lets talk about this in a little more detail.

What have these two transducers got in common? They both have diaphragms. Obviously they are different sizes, but doesn't change actual transduction mechanism. What is the consequence of having a flexible or mobile diaphragm? We have a resonant system!

What is the key difference? A loudspeaker will produce a volume velocity proportional to the voltage you supply it, where as a microphone will output a voltage proportional top the pressure on its diaphragm (at least the first type of microphone we will look at does.. other microphones types can depend on the gradient of pressure).

If we measure the radiated pressure from a loudspeaker, and everything is working correctly, we should get a flat frequency response. This is what we want from a loudspeaker. But remember, this masks a couple of things. If we were to supply a loudspeaker with a constant volume velocity we would get a $+6$ dB rise in the output. This is due to the radiation efficiency of a piston being greater at high frequencies. In order to produce a flat frequency response we need the volume velocity to fall by -6 dB per octave. This is exactly what we do for a loudspeaker. How did we do it? We stuck the system resonance as low as possible. Remember, above a resonance, we get a -6 db per octave slope. This cancels exactly with the radiation efficiency to give us a flat frequency response above the resonance frequency (below the resonance we get two $+6$ dB slopes combining to give a $+12$ dB per octave slope).

Now what about a microphone? Well for a pressure sensing microphone we will find that we don't have this rising $+6$ dB per octave slope that we got from the piston impedance function. What we have is a pressure applied over a surface area. This yields a force that drives the diaphragm. This conversion from pressure to force does not depend on frequency. What is important for a microphone however, is the transduction method. I.e. what sort of motion is the microphone sensing: displacement, velocity or acceleration? This will make a big difference on how we design our microphone.

19.1 Types of Microphones

There are two stages to a microphones operation. First we have a pressure wave forcing a diaphragm (acousto-mechanical conversion). Next we have to convert this mechanical motion into an electrical signal (mechano-electrical conversion). There are a several different ways we can do this latter part. We can use a variable capacitance, Faradays law of induction, piezo-electric materials, variable resistance, optical, etc. Different types of microphones will use different transduction mechanisms. Lets look at some examples.

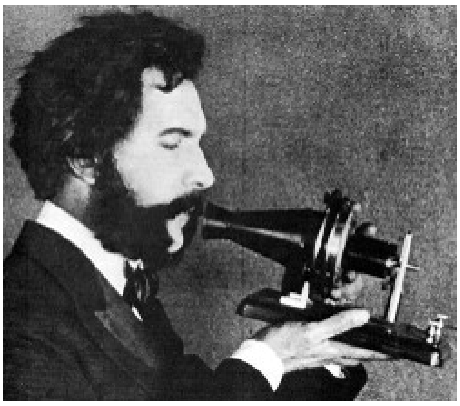


Figure 19.1: Watson.

19.1.1 Liquid Microphone

Liquid microphones were one of the first developed. The early instruments were devised and used by Bell–telephone laboratories around 1875. They were made up of a wire attached to the bottom of a parchment diaphragm. As the diaphragm was oscillated by the air, the wire was dipped in water. This water was made conductive by adding a small amount of acid. As the pressure changes caused diaphragm to move, the wire would have more or less contact with the water, thereby changing the circuit resistance. The resulting current variations in the listening device reproduce the original sounds.

According to popular legend, the 1st historic call Bell said "Mr. Watson, come here. I want to see you." Watson later recounted that Bell had spilled battery acid and had called for him over the phone with these words, but this may have been in a separate incident.

What sort of motion is this microphone sensitive to? **Displacement!**

Figure 19.2: Liquid microphone.

19.1.2 Carbon Capsule

Next up we have the carbon capsule microphone. These were developed a little later, patented by Emile Berliner in 1877. Interestingly this design was also invented by Edison and Hughs at the same time and there was a big legal patent battle between them.

The microphone was made up of two electrical contacts separated by a thin layer of carbon. With one contact attached to a diaphragm, an external pressure would cause the carbon disk to be compressed, in turn changing its resistance.

Although this design was more robust than liquid-based mic by Bell, it sounded pretty rubbish. An improved version of this was used were carbon granules were loosely packed in an enclosed space. Similarly, sound waves cause the diaphragm to vibrate. As the granules are pushed closer together their resistance decreases.

What sort of motion is this microphone sensitive to? **Displacement!**

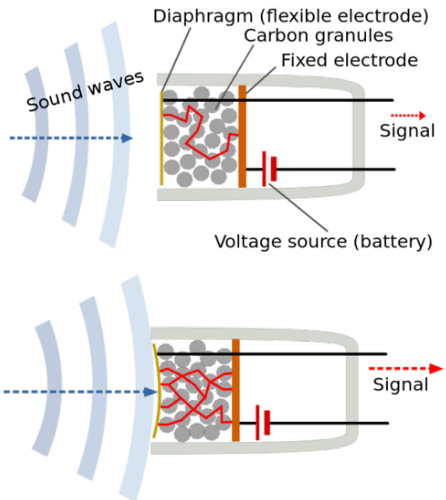


Figure 19.3: Carbon capsule microphone.

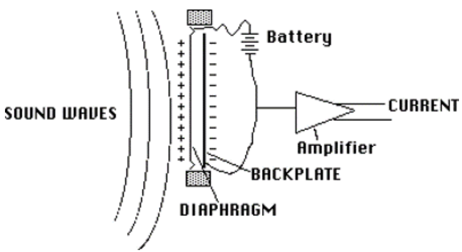
19.1.3 Capacitor Microphone

Next up we have the capacitor microphone. Now this is a design that has stood the test of time!

The first capacitor microphone was invented in 1917 by Bell Laboratories. It was initially used as a laboratory sound intensity measurement tool, although it was quickly adopted by commercial broadcasters in the early 1920s, where there was a clear need for better microphones.

The condenser microphone is essentially a variable capacitor. A capacitor is made up of two closely spaced plates biased with a polarising voltage. One of the plates either is, or has a diaphragm connected to it. When this plate moves there is a change in the capacitance/charge across the two plates. This induces a fluctuating voltage across the capacitor terminals.

What sort of motion is this sensitive to? **Displacement!**



19.1.4 Moving Coil (Electro-dynamic)

Condenser microphones were employed to a limited extend in the BBC from 1926. However, they had a reputation for being 'temperamental' due to their susceptibility to moisture causing 'frying noises'. This lead to the development of dynamic microphones as an alternative.

In a dynamic microphone (i.e. a moving coil microphone) a diaphragm is connected to conductor (i.e. some copper wire). This conductor was then placed in a permanent magnetic field. As sound waves vibrated the diaphragm the conductors movement in the magnetic field induced a voltage across its two ends (according to Faraday's law).

The first practical moving coil microphone was built in 1923. It was called 'The Marconi Skykes' or 'magnetophon'.

Electromagnetic microphones were relatively late on the scene. This was primarily because the permanent magnets of the day were very weak and only electromagnets could create sufficient flux densities. Originally the magnetic field was created by a large electromagnet consuming around 4A from an 8V battery! To block out any electromagnetic interference (which the sensitive microphone was susceptible to) the microphone was usually placed within a copper mesh box called a Faraday Cage.

What sort of motion is this microphone sensitive to? **Velocity!**

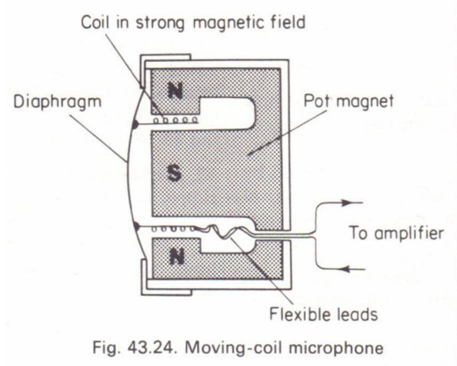


Figure 19.5: Moving coil microphone.

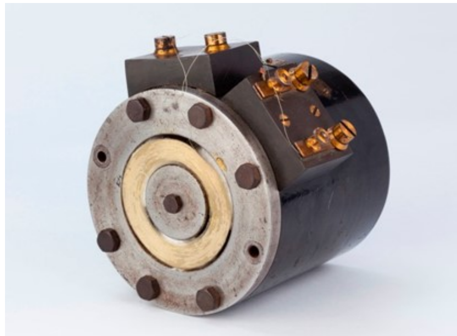


Figure 19.6: Moving coil microphone.

19.1.5 Moving Iron (Electro-dynamic)

Here is another take on the dynamic microphone, called the moving iron microphone.

In this design the diaphragm is connected to a bit of magnetised soft iron. As the diaphragm vibrates this piece of iron moves in and out of a gap in a larger magnet. This alters the magnetic flux that flows through the 'magnetic circuit'. A coil wrapped around the magnet detects this change in magnetic flux and induces a corresponding voltage.

What sort of motion does it detect? **Velocity!**

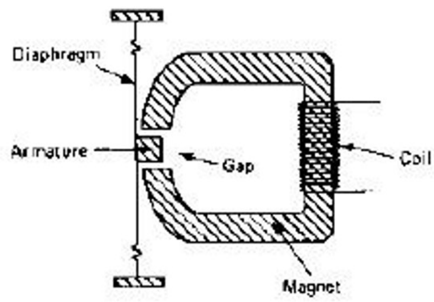


Figure 19.7: Moving iron microphone.

19.1.6 Ribbon (Electro-dynamic)

Next up we have the ribbon microphone. This is another type dynamic microphone. Rather than a a coil of wire, a ribbon microphone places a thin ribbon (i.e. a single turn in the magnetic field) in a permanent magnetic field. This microphone is believed to have been developed by Harry F. Olson, (responsible for the line array) by reverse engineering a ribbon loudspeaker.

A thin strip of aluminium foil (the ribbon) is suspended in a magnetic field. As sound waves vibrate the ribbon in the field an electrical signal is generated. The ribbon is corrugated like an accordion to reduce resonances across it. Importantly, both sides of exposed of the ribbon are exposed. For this reason, as we will see later, it has a figure of 8 type response (like a spaced dipole).

What sort of motion is the ribbon microphone sensitive to? Turns out its **acceleration!**

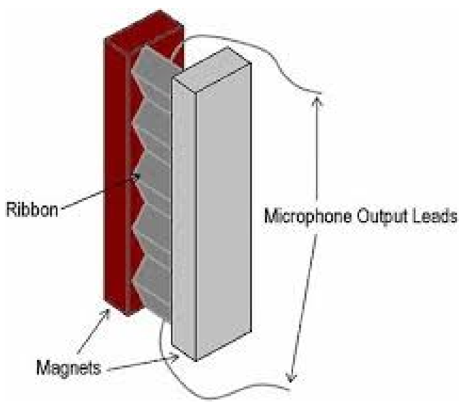


Figure 19.8: Ribbon microphone.

19.1.7 Fibre Optic

So here is a slightly more modern design, the optical microphone. How does it work? Light from a laser source travels through an optical fibre to the surface of a reflective diaphragm. Sound waves cause the diaphragm to vibrate, which in turn changes the intensity of the light it reflects. The intensity of the reflected light is then measured and transformed to an audio waveform.

The advantage of fibre optic microphones is that they do not react to or influence any electromagnetic or radioactive fields. This makes them idea for use in areas where conventional microphones are unable to operate, e.g. inside industrial turbines of MRI machines.

What sort of motion do they sense? **Displacement!**

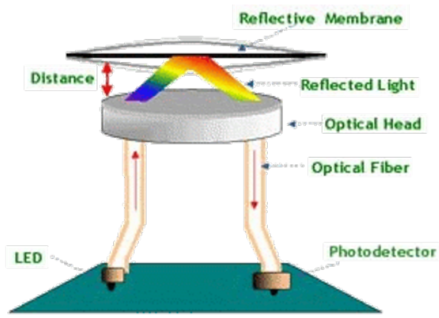


Figure 19.9: Fibre optic microphone.

19.1.8 Laser

Our final microphone is that of the laser microphone. A laser beamed onto a surface, which reflects back some portion of the incident light. The motion of the surface causes a doppler shift in the in the laser.

This doppler shift can be analysed by looking at the interference patterns that are set up in the standing wave of the laser. From this the motion of surface can be deduced.

Any idea what motion this senses? Well doppler shift is all about the velocity of a moving source, so it is velocity sensitive!

The great thing about laser microphones like this is that they can work over very long distances. The down side it that they are also very, very expensive.

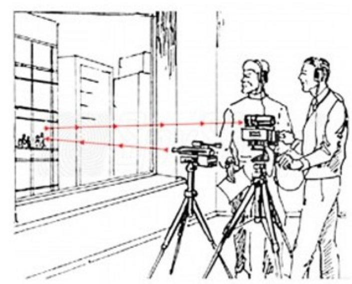


Figure 19.10: Laser microphone.

20 Mechanical Model

Lets start developing a model for the dynamics of microphone diaphragm. Irrespective of the transduction mechanism used, we can consider, as a first approximation, the diaphragm as a mass on a spring (just like our loudspeaker diaphragm).

By now you should be familiar with the dynamics of a mass spring system. We have the parameters m, r, k, and f.

As usual we start with Newton's 2nd law.

$$\sum_i F_i = m\ddot{x} \tag{20.1}$$

Substituting in the damping, spring and external forces we get an equation of motion,

$$F_{ext} - kx - r\dot{x} = m\ddot{x} \rightarrow F_{ext} = m\ddot{x} + kx + r\dot{x} \tag{20.2}$$

Next we recall that the applied force is a function of pressure at the surface and the diaphragm cross-sectional area,

$$F_{ext} = PS_d. \tag{20.3}$$

If we assume that the spring and damper are connect to something that doesn't move, the mass is free to respond to an incoming pressure wave. Just like we did with the loudspeaker, we can formulate this as an equivalent circuit. Alternatively we can go through the more mathematical approach.

Assuming the incoming pressure is periodic, the applied force and resulting motion will also be periodic, $F_{ext} = F_0 e^{j\omega t}$ and $x = x_0 e^{j\omega t}$. Substituting these into our equation of motion, and evaluating the derivatives we arrive at,

$$F_{ext} = -\omega^2 mx + kx + j\omega rx = \left(-\omega^2 m + k + j\omega r\right) x. \tag{20.4}$$

From the above we can determined the displacement response of the diaphragm (assuming a steady state response, remember there is also a transient part, although we can ignore that here),

$$x = \frac{F_{ext}}{(-\omega^2 m + k + j\omega r)}. \tag{20.5}$$

A microphone that senses displacement would have a frequency response shape that follows this equation.

Now what a about velocity sensing microphone? Its diaphragm will still behave as mass spring system, so all we need to do is find the systems velocity response. We can do this my just differentiating the displacement response. Remember, for a periodic response,

$$v = \dot{x} = \frac{d}{dt} x_0 e^{j\omega t} = j\omega x_0 e^{j\omega t} = j\omega x. \tag{20.6}$$

Differentiation with respect to time has the effect of introducing a factor of $j\omega$ in the numerator. Now by differentiating the displacement response derived above,

$$v = \dot{x} = \frac{j\omega F_{ext}}{(-\omega^2 m + k + j\omega r)}. \tag{20.7}$$

We can then rearrange this into a more convenient form,

$$v = \dot{x} = \frac{F_{ext}}{\left(j\omega m + \frac{k}{j\omega} + r\right)}. \tag{20.8}$$

A microphone that senses velocity would have a frequency response shape that follows this equation.

What about the acceleration response? Same idea. Differentiate the velocity response. This gives us another factor of $j\omega$.

$$a = \dot{v} = \ddot{x} = \frac{j\omega F_{ext}}{\left(j\omega m + \frac{k}{j\omega} + r\right)} = \frac{F_{ext}}{\left(m - \frac{k}{\omega^2} + \frac{r}{j\omega}\right)} \tag{20.9}$$

A microphone that senses acceleration would have a frequency response shape that follows this equation.

Another way of deriving the diaphragm dynamics is to formulate an equivalent circuit, just like we did for our loudspeaker model. We wont consider this approach any further here, as it is virtually identical to that of a loudspeaker diaphragm, which we have already covered.

20.0.1 Displacement Sensing Microphones

Suppose we have a displacement sensing microphone. Its frequency response is shown in figure 20.1. It has a flat region below the resonance, where the mass term is negligible. This is referred to as the stiffness controlled region. Above the resonance the mass term dominates, and since it is scaled by a factor of ω^2, we get a -12 dB per octave drop off.

From equation 20.5 we can see that in the flat region, the displacement response depends only on the stiffness, and is inversely dependent on it,

$$x = \frac{F_{ext}}{(-\omega^2 m + k + j\omega r)} \xrightarrow{\omega \to 0} \frac{F}{k}. \tag{20.10}$$

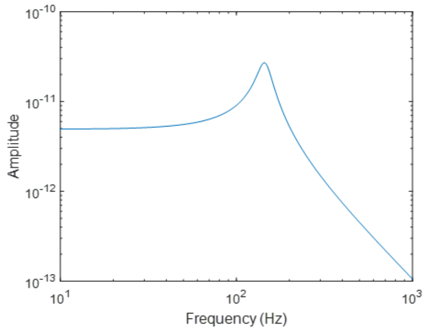


Figure 20.1: Response of displacement sensing microphone.

If our microphone senses displacement, what part of this response would we want to use? The flat bit. How could we maximise the useable bandwidth? We make the diaphragm as stiff as possible. This pushes up the resonance, extending the flat region of the frequency response. This is exactly how condenser microphones are designed.

20.0.2 Velocity Sensing Microphones

Suppose we have a velocity sensing microphone. Its frequency response is shown in figure 20.2. Note that the resonant frequency is now centred between $+6$ and -6 dB per octave slopes. We call this is a damping controlled response. By increasing the level damping the resonant peak will broaden.

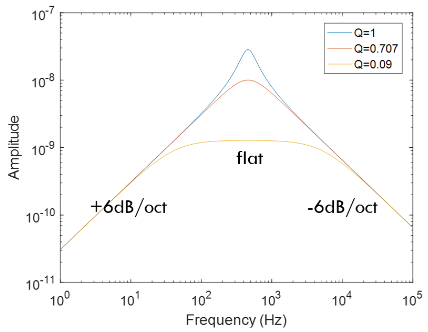


Figure 20.2: Response of velocity sensing microphone.

Suppose we want to design a velocity sensing microphone, how would we do this? Clearly the stiffness and mass controlled regions are now out of use, because of their ± 6 dB slopes. What can we do? If we add enough damping, we can actually do a pretty good job flatting out the resonant peak. Then we just have to make sure that it is placed in the centre of the audible range. Some different levels of damping are shown in figure 20.2.

To achieve a wide enough flat region, we need a Q factor much much lower than 0.707. If it were 0.707 we would actually get quite a peaky sounding microphone (like an old novelty record). To achieve a flat frequency response we need the damping to be quite extreme. As a consequence dynamic microphones are usually less sensitive that condenser microphones, of the order of 8 times less sensitive.

20.0.3 Acceleration Sensing Microphones

Suppose we have a acceleration sensing microphone. Its frequency response is shown in figure 20.3.

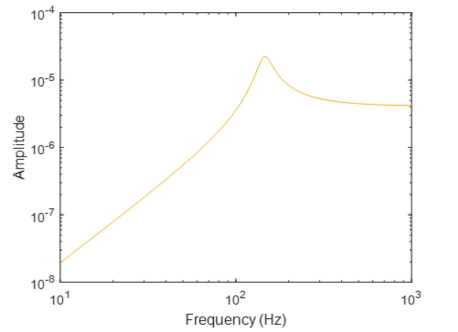


Figure 20.3: Response of acceleration sensing microphone.

Above the resonance we get a nice flat frequency response. Only the mass term has any influence in this region. We call it the mass controlled region.

$$a = \frac{F_{ext}}{\left(m - \frac{k}{\omega^2} + \frac{r}{j\omega}\right)} \xrightarrow{\omega\to\infty} \frac{F}{m} \tag{20.11}$$

At low frequencies the mass and damping terms are very small compared to stiffness term. This leads to a factor of ω^2, which gives us a $+12$ dB per octave rise. The system resonance, which occurs when the reactive part disappears $m\omega - k/\omega = 0$ is limited by the damping present. The greater the damping the low the resonance amplitude, and visa versa.

Suppose we want to design a acceleration sensing microphone. We want to maximise the useable bandwidth by putting the resonant frequency as low possible. How do we do this? Increase the mass and reduce the stiffness.

20.1 Temporal Response

What about the temporal response of a microphone? Remember, the frequency response is only part of the picture. Lets consider a displacement sensing microphone, like the capacitor/condenser microphone.

To design a capacitor microphone we have a resonance that we need to push up to the top of the band. This means we need the diaphragm to be very light and stiff. Often achieving the high stiffness is done by putting a really thin diaphragm under tension rather than through spring like stiffness. What about its damping?

One of the most important parameters when designing a microphone is the Q factor. The Q factor is related to the amount of damping present, and influences both the time and frequency response of the microphone, see for example figures 20.4 and 20.5 where the frequency and time response are shown for a number of different Q factors. We could choose a value of Q that will give us the fastest time response without oscillations, giving the least amount of smearing in time. To achieve this we would require a critical damping, giving a Q-factor of 0.5. As a consequence we would get a more narrow pass band in the frequency response, as in figure 20.4.

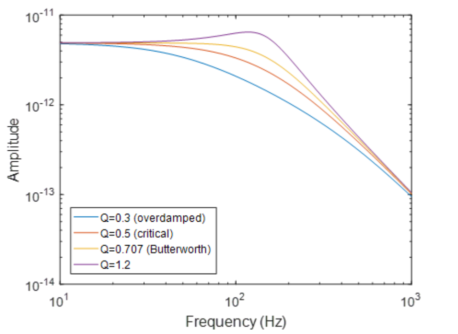
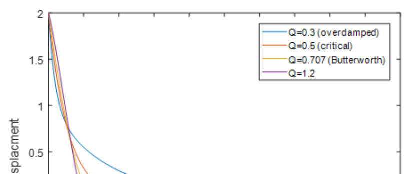


Figure 20.4: Frequency response of displacement sensing microphone for different levels of damping.

Nevertheless, this is still a better design than a Q factor which is over-damped (i.e. takes ages to decay). In general it is the frequency response that is considered more important in microphone design.

The difficulty is that the frequency response isn't that great with $Q = 0.5$. A better response would be the Butterworth alignment where we have the maximally



flat response ($Q = 0.707$). This gives the microphone a wider frequency range, at the cost of some small oscillation in the time domain.

In practice we may want a Q a little larger than 0.707, the reason being is that usually you can get away with a slightly lower resonance. It is quite hard to make something very stiff and light too.

Reducing the damping a little (raising the Q) has the effect of pushing up the level at and above the resonance. We can use this to extend the frequency range a little bit. In fact people often quite like this effect,for example in vocal recording. It tends to bring a bit of brightness or air to the vocals, which can help bring it out of the mix.

20.2 Microphone Design Summary

To design a microphone we need to know the transduction method, i.e. what motion is being sensed (i.e. acceleration, velocity, or displacement). Then it's a case of adjusting the diaphragm properties so that over the bandwidth of interest we get a flat frequency response.

When we did this with a loudspeaker we designed the system so that the resonance was at the lower end of the frequency response.

Then we end up with this -6dB per octave roll off on the velocity response which is compensated for by the impedance of a piston which has a + 6dB / octave behaviour. They cancel out.

The design strategies to achieve a microphone with a broad flat frequency response depend on whether the transduction method is displacement, velocity, or acceleration sensitive. The mass, stiffness and damping of the microphone diaphragm are used control its frequency of resonance. Its position determines whether the microphone's mass, stiffness or damping controlled regions are used.

When we did this with a loudspeaker we designed the system so that the resonance was at the lower end of the frequency response. We ended up with a -6 dB per octave roll off on the velocity response which was compensated for by the impedance of a piston which has a $+6$ dB per octave slope. They cancel out. When designing a microphone we get a few extra options, depending on the transduction method being used.

- If we are sensing **displacement** the resonance is pushed to the top end of, or above, the audio band. We do this by making the diaphragm and suspension **light weight and very stiff**.
- If we are sensing **velocity** the resonance is placed in the middle of the audio band. To achieve a useable bandwidth a **large amount of damping** is applied to flatten the resonant response.
- If we are sensing **acceleration** the resonance is pushed to the bottom end of, or below, the audio band. We do this by making the diaphragm and suspension **heavy and compliant**.

It is important to note that the design strategies described above are concerned with omni-directional microphones. Things will change a little if we have a different polar pattern, as we will see later.

21 Transduction Methods

Now that we have a working model for the dynamics of a microphone diaphragm, lets consider in a bit more detail the two most common forms of transduction: electro-dynamic, and capacitive.

21.1 Electro-dynamic

Shown in figure 21.1 is a diagram illustrating the construction of a typical electro-dynamic microphone. A diaphragm that is fastened to a coil of wire (voice coil), which is itself situated in a permanent magnetic field, just like a dynamic loudspeaker. From our loudspeaker work in the previous semester we already know that the voltage across a conducting wire in a magnetic field is proportional to its velocity, length and the magnetic flux density,

$$V = Blu. \tag{21.1}$$

The velocity of the microphone diaphragm, due to an applied pressure, can be determined using the simple mass-spring system we just derived,

$$v = \frac{F_{ext}}{\left(j\omega m + \frac{k}{j\omega} + r\right)}. \tag{21.2}$$

So what happens under operation? Sound first passes through any wind shield present, and then through an array of small holes in front of the diaphragm. A wind shield will add some acoustic resistance and mass, whilst the small holes will act mostly as an acoustic resistance. Next a silk damping screen is connected behind the diaphragm, separating the rear cavity into two sections. This resistance dominates, since the diaphragm resistance is negligible in comparison. The total damping and mass (for grid shield and silk) is expressed as M_{AS} and R_{AS}.

The acoustic mass and damping created by the wind shield/grid, along with the mass and compliance of the diaphragm, controls main resonance of the system (frequency and damping). If we combine all the damping and mass terms this is just the same as the single DoF system that we used to model loudspeakers. An equivalent circuit for the microphone in this mid frequency range is shown in figure 21.2.

What goes on behind the diaphragm is a bit more complicated. To compensate for the 6 dB per octave roll off with decreasing frequency there is a complaint volume behind the chamber (C_{AB}). A thin air tube is also added, this is connected to the outside air. The purpose of this tube is two fold. Firstly it is used to prevent static displacements of the diaphragm due to changes in atmospheric pressure. Obviously we want our microphone to work whatever the atmospheric pressure, so we want it to ignore slow changes in pressure, and only 'listen to' the acoustic

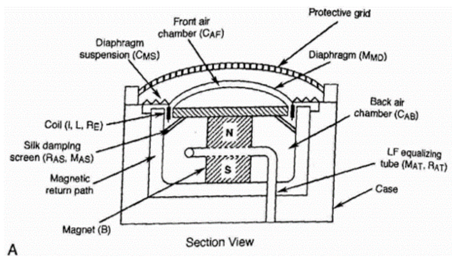


Figure 21.1: Construction fo typical dynamic microphone.

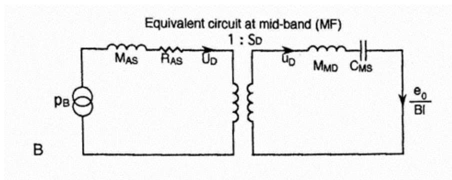


Figure 21.2: Equivalent circuit for mid frequency operation of a typical dynamic microphone.

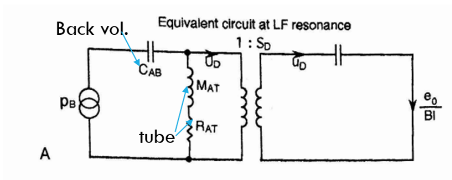


Figure 21.3: Equivalent circuit for low frequency operation of a typical dynamic microphone.

fluctuations. Secondly it is used to introduce an acoustic mass and resistance (M_{AT} and R_{AT}). The dimensions of tube are chosen so that when combined with the rear cavity compliance, we get an additional Helmholtz resonance at the lower end of the response. This is used to provide a little bit of bass boost at low frequencies, thus extending the useable bandwidth of the microphone. An equivalent circuit for the microphone in this low frequency range is shown in figure 21.3.

It turns out that at high frequencies there is another Helmholtz resonance, instead determined by the mass of the driver and the compliance of the small air chamber just behind the diaphragm (C_{AF}). This can also be tuned to give a boost, but at high frequencies, which could add a bit of brightness to vocals. An equivalent circuit for the microphone in this low frequency range is shown in figure 21.3.

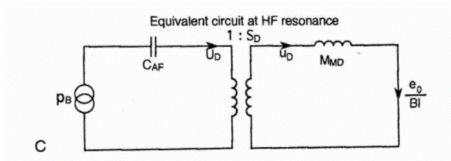


Figure 21.4: Equivalent circuit for high frequency operation of a typical dynamic microphone.

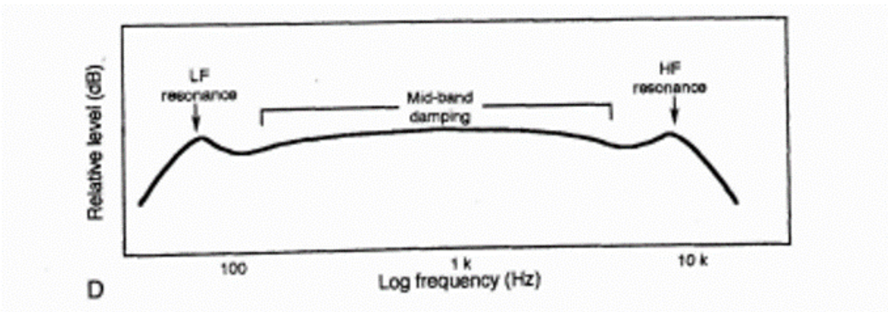


Figure 21.5: Frequency response of dynamic microphone with dual Helmholtz resonance.

All together what do we have we get the frequency response shown in figure 21.5. At low frequencies, due to the back compliant volume with a thin tube we get a Helmholtz resonance. At high frequencies the mass of the driver and the small compliant volume behind it give us a second resonance. In the mid frequency range the response is dominated by the mass spring dynamics of the diaphragm. By applying a silk screen (and front grid) a very high level of damping is achieved, and so the main resonance is flattened out into a pass-band region.

These mechanical and acoustic systems can be combined into the equivalent circuit shown in figure 21.6. There are a number of different ways to improve the response of a microphone, these are just some examples. In any case, it is just like with our loudspeaker design, the whole process is a bit of resonance jenga.

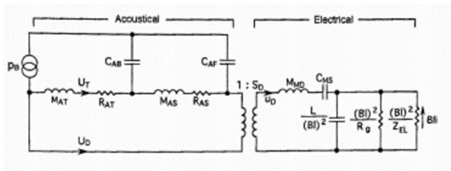


Figure 21.6: Combined equivalent circuit for the operation of a typical dynamic microphone.

21.2 Capacitance

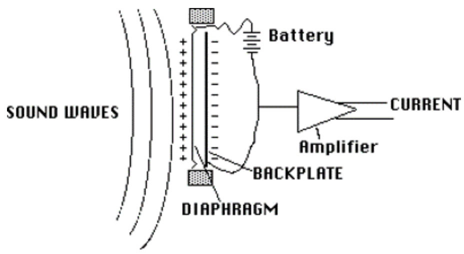


Figure 21.7: Simple diagram of capacitor microphone.

The operation of a capacitor microphone is fundamentally different from a moving coil/dynamic design. Shown in figure **??** is a simplified diagram of a capacitor microphone. A bias voltage is applied across a thin sheet of metal (diaphragm) and a back plate. Together the diaphragm and back plate form a variable capacitor. As the diaphragm is displaced by a sound wave, the capacitance changes, which super imposes a voltage fluctuation onto the bias voltage.

Shown in figures 21.8 and 21.9 are a couple schematics of a typical capacitor microphone design. We have a rigid dust cap/protective grid (very important since the diaphragm is very very delicate). Below this we have the diaphragm, which is usually a thin metal film, or a metal coated plastic. After a small air

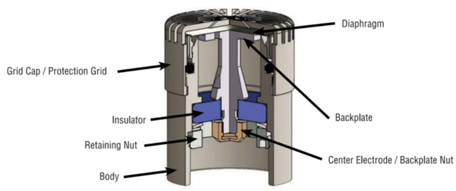


Figure 21.8: Schematic of capacitor microphone.

gap we have the back plate. This back plate must be electrically isolated from the rest of the capsule, else the diaphragm wont function as a variable capacitor. Behind the back plate we have a small capillary tube, its purpose being the same as a dynamic microphone, to prevent static displacements due to atmospheric pressure changes.

An important requirement of a capacitor microphone is that a pre-amplifier is located very close to the capsule. This is because any cable attached will have its own capacitance. This would appear in parallel with the diaphragm capacitance and so we would get a reduction in the over all impedance and sensitivity of the microphone, hence increasing its susceptibility to noise. It is for this reason that all capacitor microphones must have their own preamp, or some buffer nearby.

So what's the maths like? Well its not very fun. But if we make a few not so drastic assumptions, we can make life a lot easier for ourselves.

We can model a capacitor microphone using the simple circuit shown in figure **??**. A bias voltage is applied across a series resistor and capacitor (the diaphragm), over which the output voltage is taken. The purpose of the resistor is to generate a large potential across the capacitor.

The voltage across a capacitor is given by the charge divided by the capacitance,

$$V = \frac{Q}{C} \tag{21.3}$$

If we think of our diaphragm and back plates as a pair of plates, we can calculate the capacitance straight forwardly. It is proportional to the surface area, and inversely proportional to the plate separation,

$$C = \frac{\epsilon S}{X}. \tag{21.4}$$

Taking each quantity and separating them into a static part and dynamic part we have,

$$V = V_0 + v, \quad I = I_0 + i, \quad Q = Q_0 + q, \quad X = X_0 + x. \tag{21.5}$$

Now lets consider the voltage across the capacitor (i.e. the diaphragm). It will have a constant and dynamic part, as will the charge and distance terms,

$$V_0 + v = \frac{Q}{C} = \frac{QX}{\epsilon S} = \frac{(Q_0 + q)(X_0 + x)}{\epsilon S}. \tag{21.6}$$

If we consider the case in which no sound in present, i.e. the dynamic part is 0, then the bias voltage can be found.

$$V_0 = \frac{Q_0 X_0}{\epsilon S} \tag{21.7}$$

Now lets multiply out the brackets in equation 21.6.

$$V_0 + v = \frac{Q_0 X_0}{\epsilon S} + \frac{Q_0 x}{\epsilon S} + \frac{q + X_0}{\epsilon S} + \frac{q + x}{\epsilon S}. \tag{21.8}$$

Assuming that the dynamic part is small compared to the constant part, we can neglect any second order terms. This leads to a linear equation for the dynamic voltage.

$$v = \frac{Q_0 x}{\epsilon S} + \frac{q + X_0}{\epsilon S}. \tag{21.9}$$

Finally, if we can assume that the charge across the plate is constant (i.e. $q = 0$) we arrive at a nice simple expression for the voltage output of a capacitor microphone,

$$v = \frac{Q_0 x}{\epsilon S}. \tag{21.10}$$

An approx. constant charge can be obtained by either:

1. A high DC polarising voltage (order $100V$) applied through a very high impedance (a resistor of order GΩs), or

2. Embedding charge (at the time of manufacture) in a very high resistivity ('electret') film.

This ensures an essentially constant charge on the microphone, even when its capacitance changes due to the sound pressure on its diaphragm. When one of the plates is moved this changes the capacitance and due to the constant charge leads to an AC-voltage across the plates representing the displacement. The AC signal is finally separated from the polarising voltage via a series capacitor.

We have all the pieces. We know the dynamic voltage, which depends on the bias voltage (which we also know), and we know the mechanical behaviour of the diaphragm (a mass spring system). So lets put it all together. Substituting in the bias voltage and the diaphragm displacement,

$$v = \frac{Q_0}{\epsilon S} \frac{pS}{\left(j\omega R - \omega^2 M + \frac{1}{C_M}\right)}. \tag{21.11}$$

which after a little bit of rearrangement becomes,

$$v = \frac{Q_0 p C_M}{\epsilon} \frac{1}{(j\omega C_M R - \omega^2 C_M M + 1)}. \tag{21.12}$$

Finally we re-parametrized this equation by using diaphragm resonance ω_c and Q factor,

$$v = \frac{Q_0 p C_M}{\epsilon} \frac{1}{\left(1 + \frac{1}{Q_{TS}} \frac{j\omega}{\omega_c} + \left(\frac{j\omega}{\omega_c}\right)^2\right)}. \tag{21.13}$$

This is our dynamic output voltage.

In the operating regime (assuming a constant current) the microphones sensitivity is given by the term up front,

$$\frac{v}{p} = \frac{Q_0 C_M}{\epsilon} \tag{21.14}$$

It depends on the mechanical compliance C_M, the charge across the capacitor Q_0, and the constant ϵ.

So in deriving this equation we neglected some second order terms. These terms can be thought of as introducing distortion into the signal. These distortions are generally small so long as the dynamic variables are small relative to the static ones.

We also assumed that the dynamic part of the charge was negligible. How do we ensure a constant charge across the microphone diaphragm? Well we have two options. We can either using a high DC polarising voltage, applied through a very large impedance, or we can embed the charge at the time of manufacturer using a very high resistivity film.

Microphones that use this second approach, where the charge is embedded, are called *electret* microphones. Their advantage is that the embedded charge eliminates the need for a polarizing power supply, by instead using a permanently charged material. These microphone types normally contain an integrated preamplifier, which does require a small amount of power (often incorrectly called polarizing power or bias). Downside, these microphones only have a limited life span as charge decays over about 30-40 years.

22 Velocity Sensing

Up until now we have been focusing on microphones that sense, or measure, acoustic pressure. But we know that there is more to an acoustic wave than just pressure, there is also particle velocity (remember, they are related by Euler's equation).

Pressure is a scalar quantity that tells us how bunched up, or stretched apart, the particles of air are. Particle velocity on the other hand tells us how fast, and in what direction the air particles are moving. Acoustic pressure nd particle velocity are related by the properties of the medium. Their exact relation depends also on the type of wave. We are interested in developing microphones that can sense (or measure) particle velocity, as opposed to pressure.

How do we do this? There a few different ways. We will talk a bit about the more exotic approaches before we settle on a conventional method.

22.1 Transduction Methods

22.1.1 Laser Doppler Velocimeter

The first exotic approach is based on the doppler shift in the light reflected by moving particles suspended in the air. It is exotic because it uses lasers... A schematic of the experimental set up is shown in figure 22.1.

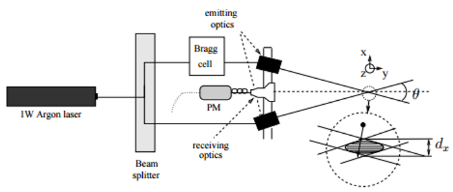


Figure 22.1: Schematic of laser doppler velocimeter microphone setup

Two laser beams of equal intensity are crossed and focused at the point under investigation. Together these lasers form an ellipsoidal volume consisting dark and bright fringes due to the interference between the two beams. Positioned next to the lasers is an optical receiver. This records the scattered light from the volume. To get some light scattering we have to fill the volume with particles (tracer). In some cases there may be enough natural particulate in the air. Now when the fluid moves (with a given particle velocity) the scattered light from the particles is Doppler shifted. This Doppler shift is directly related to the motion of fluid.

Why all the effort? Laser velocimeters are essential non-intrusive (normal microphone disturb the sound field they are trying to measure!). This makes them suitable for hostile environments, and very accurate. Another bonus is that no calibration is required, and we can get a good spatial and temporal resolution by scanning the two beams.

Downside? Obviously these things aren't cheap. Also the flow must be seeded with some particulate if none naturally exist. Also, it's a single point method, so we cant measure at multiple points simultaneously.

22.1.2 Hot Wire Anemometer

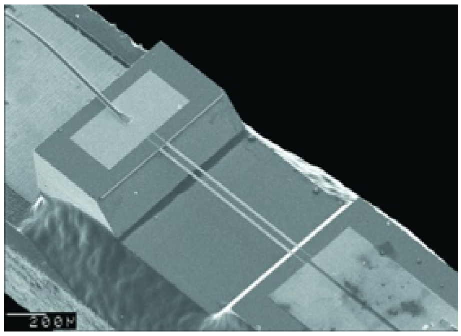


Figure 22.2: Close up image of hot wire anemometer.

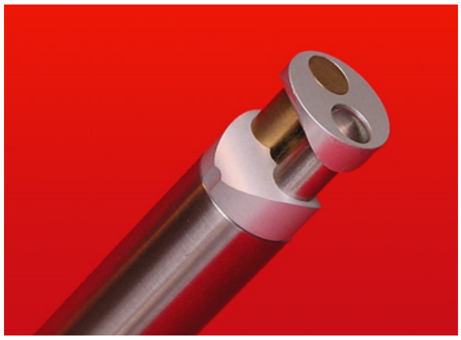


Figure 22.3: Microflown hot wire anemometer.

Another exotic approach, albeit more affordable, is the hot wire anemometer.

Here is the idea. If you run a current through two very thin wires (see figure 22.2), they will both heat up. As air moves over the wires, they are cooled down. This change in temperature changes the resistance of the wire and so we can record the resulting voltage drop. It has two wires, which when subject to a sound wave, asymmetrically alter their temperature distribution thus resistance. The upstream sensor is heated by the downstream sensor. This lets us distinguish between positive and negative velocity direction. The temperature difference of the two sensors quantifies the particle velocity for a particular axis.

A commercially available hot wire anemometer, called a microflown, is shown in figure 22.3.

Since each sensor is made up of two wires, these microphones give us a figure of 8 directivity pattern. We can use multiple sensors to record particle velocity in 2D and 3D.

Its it worth noting that these things aren't the worlds best microphone. You get quite a lot of noise. Their purpose is more for measurement than recording. An example application might be to measure an acoustic impedance (need pressure and particle velocity).

22.1.3 Pressure Gradient Microphone

Now its time for a practical approach. Pressure gradient microphones.

Remember, according to Euler's equation,

$$\frac{du}{dx} = \frac{1}{\rho}\frac{dp}{dx} \tag{22.1}$$

the spatial derivative of pressure is related to the temporal derivative of particle velocity. So if we can measure the pressure gradient (i.e. its spatial derivative) then we should get something proportional to particle velocity.

There are two common ways of measuring a pressure gradient. The first is to expose both sides of a diaphragm to the incident sound, the other is to use two separate diaphragms altogether. Lets consider the first approach.

Thus far we have been considering microphones that are sensitive to pressure. A typical example is shown in figure 22.4, where a diaphragm is placed at one end of a closed tube. The total pressure transduced by this diaphragm is simply the pressure to its front side,

$$p_T = p_f. \tag{22.2}$$

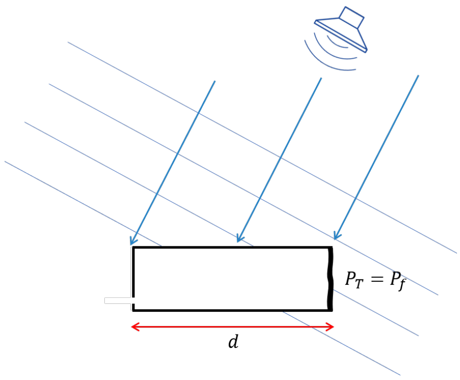


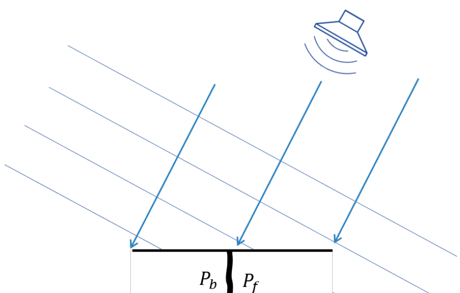
Figure 22.4: Diaphragm placed at one end of a closed tube (small opening for pressure equalisation.

Notice that there is no frequency or angular dependence to this transduced pressure. This means that we get an omni-directional response (pressure is sensed equally from all directions).

Now suppose we place the diaphragm in the centre of the tube and expose its rear side, as in figure 22.5. The movement of the diaphragm will be determined by the total pressure acting on it. This is now given by the difference between the front and rear pressures,

$$p_T = p_f - p_b \tag{22.3}$$

If the pressure is the same on the front and back, there is no net force and the diaphragm wont move. If there is difference in the pressure (i.e. a gradient) then the diaphragm will be displaced.



Note that we can express the rear pressure P_b in terms of the front pressure P_f as,

$$p_b = p_f + \frac{dp}{dx}\Delta x. \tag{22.4}$$

This is true only when P_f and P_b are spaced infinitesimally close. It is however approximately true for small (but not infinitesimal) spacings. Rearranging the above the pressure gradient if given approximately by,

$$\frac{dp}{dx} \approx \frac{p_b - p_f}{\Delta x}. \tag{22.5}$$

Notice that this is equivalent to,

$$\frac{dp}{dx} \approx -\frac{p_T}{\Delta x} \tag{22.6}$$

where p_T is the total pressure acting on the diaphragm. Substituting this result into Euler's equation, and integrating with respect to time, the particle velocity is given by,

$$u \approx -\frac{1}{\rho}\int \frac{p_T}{\Delta x} dt. \tag{22.7}$$

From the above we can see that the particle velocity is proportional to the total pressure on the diaphragm.

22.2 Microphone Design

We have seen that a diaphragm placed in a tube, with its rear also exposed, will be acted on by a total pressure that is proportional to the particle velocity (i.e. a pressure gradient). Now lets think a little more carefully about our diaphragm and what exactly these pressures are.

What is the difference between the front and rear pressure? There is clearly a phase difference due to the extra path length required for sound wave to reach the rear opening of the tube. It turns out this is exactly the same problem as a diaphragm in a baffle.

Lets derive the total pressure on the diaphragm. The pressure on the front is given by,

$$p_f = p_0 e^{j(\omega t - k(r+d/2))} \tag{22.8}$$

where the factor of $d/2$ accounts for the position of the diaphragm within the tube. The pressure on the rear is given similarly by,

$$p_b = p_0 e^{j(\omega t - k(r+d/2))} e^{jkd\cos\theta} = P_f e^{jkd\cos\theta} \tag{22.9}$$

where the factor $d\cos\theta$ accounts for the additional path length travelled outside the tube (this is exactly the same as our dipole loudspeaker configuration). The total pressure is then given by,

$$p_T = p_f - p_b = p_f\left(1 + e^{jkd\cos\theta}\right). \tag{22.10}$$

Following the same steps as with the dipole loudspeaker (Euler's formula, and low frequency assumption $kd << 1$) we obtain the total pressure,

$$p_T = p_f - p_b = jkdp_f\cos\theta. \tag{22.11}$$

It is useful to split this equation into two parts. The first bit contains the frequency dependent part ($jkdP_f$) and the second part accounts for the angular dependence ($\cos\theta$).

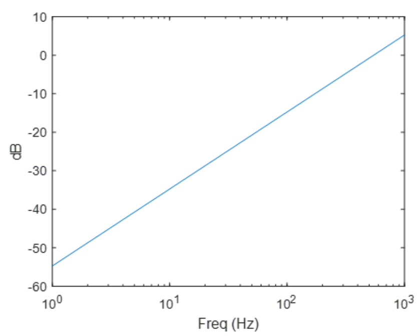


Figure 22.6: Frequency dependence of total pressure/pressure gradient.

The directivity is determined by this $\cos$ term. As you might expect this gives us the so called figure of 8 polar pattern shown in figure 22.7. This means towards the front and rear you have a large response, whilst at 90 degrees you get nothing. It is important to remember though, the front and rear are 180 degrees out of phase with one another.

Perhaps more important is frequency dependent term $jkdP_f$. A consequence of making a microphone directional, i.e. by sensing pressure gradient, is the introduction of an upwards sloping frequency response, compared to the constant response of a pressure sensing microphone. We saw a similar effect with out dipole loudspeaker.

This upwards sloping response is very interesting, particularly in the context of transduction. Remember, the transduction mechanism and diaphragm dynamics are closely related. If we change the diaphragm dynamics (i.e. by making it directional) we should probably think about what is being transduced.

By making the microphone sensitive to a pressure gradient we have essentially introduced a 6 dB per octave rise to the frequency response. To achieve a flat microphone response we will have to alter the diaphragm dynamics to compensate, i.e. we will have to change where we put the main resonance.

Suppose we have a microphone with flat pressure response, say a condenser microphone at low frequencies. Its displacement response is given by,

$$x = \frac{p_T S}{\left(j\omega R - \omega^2 M + \frac{1}{C_M}\right)} \tag{22.12}$$

where p_T is the total pressure. For a microphone with a closed rear, the total pressure is simply the pressure on the front of the diaphragm, $p_T = p_f$. If we open up the rear of the microphone, the total pressure becomes that of equation 22.11. The displacement response then becomes (ignoring the angular dependence),

$$x = \frac{jkdp_f S}{\left(j\omega R - \omega^2 M + \frac{1}{C_M}\right)}. \tag{22.13}$$

Substituting $k = \omega/c$ and after some simple manipulations we get,

$$x = \frac{d}{c}\frac{p_f S}{R + j\omega\left(\omega M + \frac{k}{\omega}\right)}. \tag{22.14}$$

This equation is nearly identical (except for the factor of d/c) to the velocity response we derived in equation 20.8. By sensing a pressure gradient the displacement response of our diaphragm no longer follows the expected shape (flat at low frequencies). It follows the resonant shape of a velocity response (even though its still displacement). This means that our original design strategy for a displacement sensing microphone (i.e. pushing the resonance as high as possible) is no longer appropriate.

What about a velocity sensing microphone? We get a similar effect. The frequency dependence introduced by a pressure gradient gives a velocity response that looks like an acceleration response. So our design strategy here also needs reconsidering.

Lets reconsider our design strategies for pressure gradient microphones.

22.2.1 Pressure Gradient Condenser Microphone

Lets start with the problem of a condenser microphone. If you remember our original design strategy to achieve a flat frequency response was to push the resonant frequency up as high as possible. This gave us a broadband flat response over the stiffness controlled region. This strategy was based on a total pressure that had no frequency dependence.

Now suppose we convert the microphone into a directional one by letting some sound come in through the back. With reference to figure 22.8, we go from having a nice flat response (blue), to a horrible resonant response (orange). This would sound pretty terrible, very tinny!

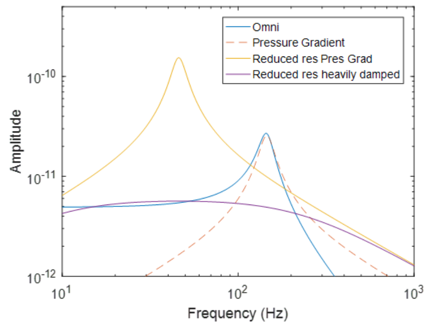


Figure 22.8: Design strategy for pressure gradient condenser microphone.

The effect of sensing a pressure gradient is akin to rotating the frequency response plot anticlockwise. Now we no longer have a flat region. How do fix this problem? What can we do to get a flat frequency response now? We need to move the resonance down (yellow), and apply lots of damping (purple)!

This is quite an easy fix. To move the resonance down we just have to reduce the diaphragm stiffness. Then we just need to add a sufficient amount of damping. How might we do this? You could put a damping layer on the diaphragm (not good idea due to increased mass). Instead some damping material is usually placed behind the diaphragm, e.g. a thin plate with lots of holes in it. This adds friction and resistive losses.

You might expect all this damping to reduce the sensitivity, which it does, though less than you expect. Since we have reduced the stiffness, the diaphragm is free to move more, which helps. Nevertheless, you typically find that the self noise of a directional condenser mic is greater than omnidirectional one.

22.2.2 Pressure Gradient Dynamic Microphone

What about electro-dynamic microphones? Remember, dynamic microphones transduce velocity. For an omni-directional (i.e. no rear opening, $P_T = P_f$) our original design strategy to achieve a flat frequency response was to place the resonant frequency in the middle of the target bandwidth, and apply loads and loads of damping to flatten out the main resonance.

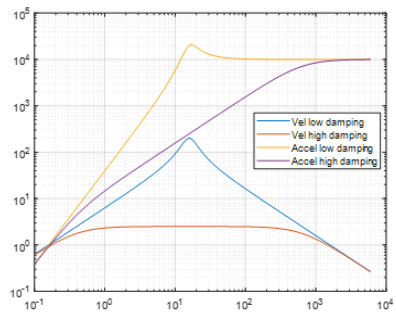


Figure 22.9: Design strategy for pressure gradient dynamic microphone.

What happens if we try to sense pressure gradient? We introduce a 6 dB per octave rise (i.e. we rotate our frequency response a little anti-clockwise). With reference to figure **??**, we go from having a nice flat response (orange), to a quite od looking response (purple). At low frequencies, where we had $+6$ dB/Oct we get $+12$ dB/Oct. In the middle where it was flat, we get $+6$dB/Oct. Now, quite surprisingly, we get a flat upper region. This flat upper region is the part of the response that we want to try and use. How do we do this?

We need to move the resonant frequency as low as possible, and reduce the amount of damping applied (yellow). A Q-factor around 0.707 would be sensible. Perhaps a little larger if you wanted to extend the bass response.

These new characteristics actually play to the strengths of the electro-dynamic microphone. Their diaphragms tend to be heavy (they have a voice coil attached) and thus they naturally have a low resonance frequency. It is very difficult to make a dynamic microphone diaphragm very light, making it hard to get a resonance in the centre of the audio bandwidth (although not impossible). And then you have the damp it loads.

Figure 22.10: The almighty SM58.

Where as for a directional microphone (i.e. pressure gradient), you don't have these problems. We want the diaphragm to be heavy. This means we can make it really robust. Also, we need much less damping. A good example is the SM58 (see figure 22.10). This directional dynamic microphone has been abused for many years, and generally keeps working!

22.2.3 Pressure Gradient Design Summary

Lets recap. Depending on what you are trying to sense, i.e. pressure or pressure gradient, and how you transduce it, i.e. through displacement or velocity, the dynamics of your diaphragm must change. Below are the key design strategies for omni-directional and pressure gradient microphones are summarized below.

Condenser Microphone (displacement sensing):

- *Omni-directional* ($p_T = p_f$) - the resonance is pushed to the top end of, or above, the audio band. We do this by making the diaphragm and suspension **light weight and very stiff**.
- *Pressure gradient* ($p_T = jkdp_f$) - the resonance is placed in the centre of the audio band, and a **large amount of damping** is applied.

Dynamic Microphone (velocity sensing):

- *Omni-directional* ($p_T = p_f$) - the resonance is placed in the centre of the audio band, and a **large amount of damping** is applied.
- *Pressure gradient* ($p_T = jkdp_f$) - the resonance is pushed to the bottom end of, or below, the audio band. We do this by making the diaphragm and suspension **heavy and compliant**.

There are quite a range of different considerations when it comes to microphones, especially compared with loudspeaker design where a low frequency resonance was always best. One of the key challenges for microphone designers is to find materials that let you do what you want. For example, condensers didn't really become practical until light materials that could be put under a lot of tension where invented, for example Milar. Early measurement microphones actually used to stretch very thin aluminium for the diaphragm (these were very easy to break!).

22.3 Proximity Effect

We have seen that directional microphones differ from omni-directional ones in terms of directionality (obviously) and frequency response. This is all because we are sensing a pressure gradient, instead of pressure. Does pressure gradient sensing do anything else that is different? (Obviously it must, otherwise why would I ask).

It introduces an effect known as the 'proximity effect'. Why the *proximity* effect? Because its an effect that only occurs when the microphone is in close proximity to the source. What is the proximity effect? It translates into a bass lift as you get closer to the source. Why does this happen? It is all to do with the fact we have curved wavefronts rather than plane waves. Lets have a look and see why. Suppose we have an omni-directional source, and located near it

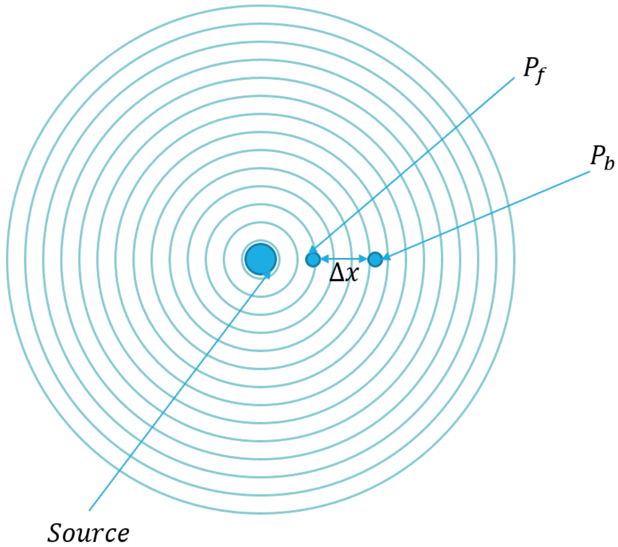


Figure 22.11: Proximity effect.

we have a pressure gradient microphone. As we move away from the source the radiated pressure falls according to $1/r$. This means the pressure on the front and rear of the diaphragm will be slightly different. This will effect the pressure gradient forcing the diaphragm. Now what happens if we move really far away? The change due to $1/r$ between the front and rear of the diaphragm decreases. At long distances the radiated pressure doesn't change much, i.e. for a plane wave pressure doesn't decrease with distance. Whereas near to the source, where we have a curved wave front, it changes a lot. This rapid change in level due to and we start to see the influence of the $1/r$ variation. Lets go through the maths and see that this really does happen.

The pressure due to a plane wave is given by,

$$p_p = Ae^{j(\omega t - kr)} \tag{22.15}$$

where A is an arbitrary amplitude coefficient. The pressure to to a spherical wave is given by,

$$p_s = \frac{A}{r}e^{j(\omega t - kr)} \tag{22.16}$$

where a factor of $1/r$ has been introduced to account for spherical spreading. Now lets consider how a pressure gradient microphone would behave in the presence of these two different wave types.

For a pressure gradient microphone the total transducer pressure is given by the front/rear pressure difference, which itself is proportional to the spatial derivative of the incoming pressure wave,

$$p_T = p_f - p_b \approx -\frac{dp}{dr}d. \tag{22.17}$$

We want to compare the transduced pressure in the presence of plane and spherical waves, so lets start by taking the derivative of the plane wave,

$$-\frac{dp_p}{dr}d = -\frac{d}{dr}\left(Ae^{j(\omega t - kr)}\right)d = jkdAe^{j(\omega t - kr)}. \tag{22.18}$$

Note that the term $Ae^{j(\omega t-kr)}$ simply corresponds to the pressure on the front of the diaphragm, P_f, in the presence of a plane wave. So for a plane wave we have the total transduced pressure,

$$p_{T_p} = jkdp_f. \tag{22.19}$$

Lets now consider a spherical wave. Taking its spatial derivative we have,

$$-\frac{dp_s}{dr}d = -\frac{d}{dr}\left[\frac{A}{r}e^{j(\omega t-kr)}\right]d. \tag{22.20}$$

To evaluate this derivative we need to use the product rule $(fg)' = f'g + fg'$. Doing so yields,

$$-\frac{dp_s}{dr}d = -Ae^{j(\omega t-kr)}\left[-\frac{1}{r^2} - \frac{jk}{r}\right]d. \tag{22.21}$$

The factor of $-1/r^2$ comes from the derivative of $1/r$, and the factor of $-jk/r$ from the derivative of e^{-jkr}. Factoring out $1/r$ gives us a spherical pressure term outside the bracket,

$$-\frac{dp_s}{dr}d = -\frac{A}{r}e^{j(\omega t-kr)}\left[-\frac{1}{r} - jk\right]d. \tag{22.22}$$

Factoring out jk and cancelling the signs finally yields,

$$-\frac{dp_s}{dr}d = jk\frac{A}{r}e^{j(\omega t-kr)}\left[\frac{1}{jkr} + 1\right]d. \tag{22.23}$$

The pressure term $\frac{A}{r}e^{j(\omega t-kr)}$ simply represents the pressure on the front of the diaphragm P_f, in the presence of spherical wave. So for a spherical wave we have the total transduced pressure,

$$p_{T_s} = jkdp_f\left[\frac{1}{jkr} + 1\right]. \tag{22.24}$$

Note that the specific value of P_f is not important here, as we are interested in the transduced pressure for plane and spherical waves, given the same front pressure.

Now that we have the transduced pressures for plane and spherical waves lets look at their magnitude ratio,

$$R = \left|\frac{p_{T_s}}{p_{T_p}}\right|. \tag{22.25}$$

A value of $R = 1$ would indicate that the two wave pressures are sensed equally. A ratio larger than 1 would indicate that a spherical wave is sensed more efficiently than a plane wave. Substituting in the transduced pressures,

$$R = \left|\frac{jkdp_f\left[\frac{1}{jkr} + 1\right]d}{jkdp_f}\right| = \left|\frac{1}{jkr} + 1\right| \tag{22.26}$$

we arrive at the main results,

$$R = \sqrt{1 + \frac{1}{k^2r^2}} = \sqrt{1 + \frac{c^2}{\omega^2r^2}}. \tag{22.27}$$

From equation 22.27 it is clear that the spherical to plane wave ratio is greater than one, i.e. a pressure gradient microphones are more sensitive to spherical wave fronts.

At high frequencies or long distances the second term in the square root of equation 22.27 becomes negligible and the ratio tends to 1,

$$R \xrightarrow{\omega, r \to \infty} 1. \tag{22.28}$$

But at low frequencies, or very short distances, this term is much greater than one, and we get a ratio that is approximately inversely proportional to frequency and distance,

$$R \xrightarrow{\omega, r \to 0} \frac{c}{\omega r}. \tag{22.29}$$

Shown in figure 22.12 is the exact trend of the proximity effect. At high frequencies it tends to one. At low frequencies we get a linearly increasing (as frequency goes down) response. For a given source distance r, we get a rise at low frequencies. We get -6 dB/Oct boost because of this property. Also, closer to the microphone we get, the more the lower frequencies are boosted. We can arbitrary define a cut-off frequency, below which we get a 6 dB per/Oct effect, and above which we get approx. no benefit,

$$f_c = \frac{c}{2\pi r} \tag{22.30}$$

This is a common effect on handheld microphones, such as SM58 (see for example figure 22.13).

If we know in advance that a microphone will be placed really close to a source, we can actually account for this proximity effect in the microphone design. If we know we are going to get a 6 dB per/Oct boost at low frequencies we can use an electronic EQ to compensate, or perhaps we could design the microphone diaphragm (and other components) to provide the necessary cut. Consider for example the dynamic microphone response in figure **??**. A pressure sensing dynamic microphone will have a frequency response according to the diaphragm velocity (orange). A large amount of damping is applied to obtain a useable bandwidth. Suppose the rear of the microphone's diaphragm is exposed, this will shift the response shape to that of an acceleration (yellow), with a 12 dB/Oct slope, followed by a 6 dB/oct slope. If the microphone is to be positioned at a known distance, the system resonance can be altered such that the 6 dB/Oct proximity boost, compensates for the 6 dB/oct cut in the diaphragm response (green). This actually leads to a really useful effect.

By compensating for proximity effect to get flat frequency response in the design, we are ensuring flat frequency response for a certain source distance. If you can control the distance, like the commentators microphone in figure 22.15 does, then you ensure the commentators voice is flat with the help of the proximity effect; as we are transducing a curved wavefront we get the 6dB/oct boost with decreasing frequency.

This means that any plane waves are not boosted by the proximity effect. Hence, relative to the near source these are attenuated (by up to 30dB at low frequencies). This is a really good way to reject distance sources, and maximise the sound of the intended source.

An example of a microphone that uses this technique is the Coles 4104B Lip Commentator's Noise Cancelling Ribbon Microphone in figure 22.15. It can reproduce high quality commentary speech from noisy surroundings, cancelling out a considerable degree of background noise. Also, its bi directional polar response allows for one or more commentators to make clear individual commentary close side by side.

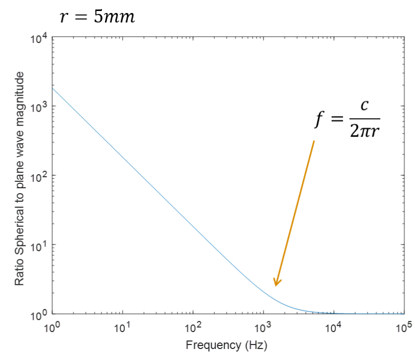


Figure 22.12: Ratio of transduced pressure for spherical and plane waves.

Figure 22.13: Utilisation of the proximity effect.

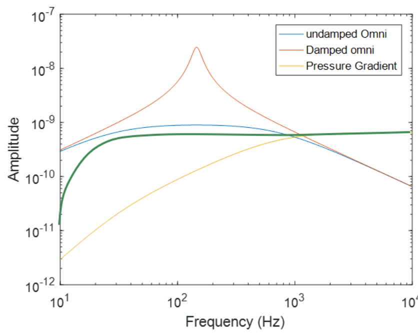


Figure 22.14: Dynamic microphone accounting for the proximity effect.

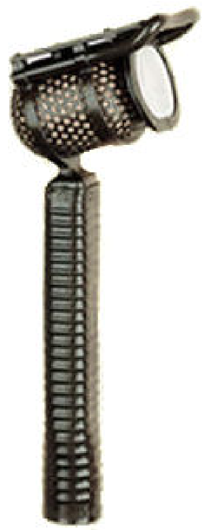


Figure 22.15: Coles 4104B Lip commentator's Noise Cancelling Ribbon Microphone

23 Diaphragm Directivity

So far we have considered some aspects of directivity, for example the effect of a sensing pressure gradient and the figure of 8 polar pattern that arises. We also looked at the proximity effect. But lets just go back and think about omnidirectional microphones again. An omnidirectional microphone senses pressure using just one side of the diaphragm. In theory it has no directional components to it. Reality is a little different.

Our omni-directional microphones have so far been based on the use infinitesimally small (point-like) diaphragms. In reality however, microphone diaphragms have some finite size. This finite diaphragm size introduces two effects, both of which contribute to the non-omni-directionality of a microphone. The first is related to phase variations over the diaphragm. The second is to do with the diffraction around/reflection off of the microphone housing. Lets start with the phase change over the diaphragm.

23.1 Phase Shift

If a sound wave, whose propagation direction is normal towards the microphone, impinges on a diaphragm all parts of the diaphragm are acted upon by the same force. I.e. the sound wave is in phase across the length of the diaphragm, as in figure 23.1 (blue). This will be true for any frequency wave, so long as it is normal incidence (assuming plane wave propagation).

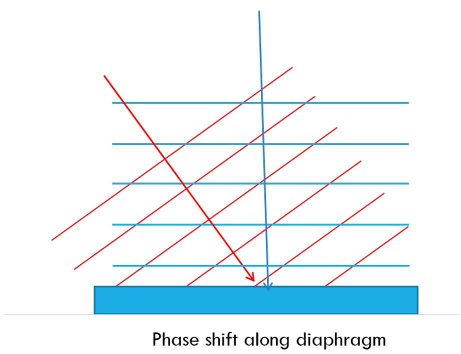


Figure 23.1: Normal and off-axis wave front over microphone diaphragm

Now suppose the same wave arrives but from some off-axis angle (red). Now there is a phase shift across the diaphragm. Different parts of the diaphragm are being acted on by different phases of the sound wave. Clearly when the wavelength is long relative to the microphone's diaphragm, this has less of an effect.

We already know that a continuous line array has a directivity pattern that is related to it's length; a long array has a very narrow beam, while a short array is more omni directional. It turns out that we get a very similar effect with microphones. Lets go through the maths and have a look why.

Lets consider the pressure across a diaphragm due to a wave coming in at some angle θ, as in figure 23.2. Taking the bottom of the diaphragm as the origin we can express the additional path length δr in terms of the angle θ and the position up the diaphragm y,

$$\Delta r = y \sin \theta. \tag{23.1}$$

Substituting this path length difference into the expression for pressure,

$$p(y) = p_0 e^{j(\omega t - k(r+\Delta r))} = P_0 e^{j(\omega t - k(r + y \sin\theta))} \tag{23.2}$$

and separating out the angular dependence we get,

$$p(y) = p_0 e^{j(\omega t - kr)} e^{-jky\sin\theta}. \tag{23.3}$$

The total pressure sensed by a diaphragm will be the average pressure across the diaphragm,

$$p_T = \frac{1}{d}\int_0^d p(y)dy = p_0 e^{j(\omega t - kr)} \frac{1}{d}\int_0^d e^{-jky\sin\theta} dy. \tag{23.4}$$

We can define a directivity factor $D(\theta)$ as the integral of the angular dependent part across the length of the diaphragm, divided by its length,

$$D(\theta) = \frac{1}{d}\int_0^d e^{-jky\sin\theta} dy. \tag{23.5}$$

Evaluating this integral over the diaphragm length we get,

$$D(\theta) = \frac{1}{d}\left[\frac{e^{-jky\sin\theta}}{-jk\sin\theta}\right]_0^d \tag{23.6}$$

and then,

$$D(\theta) = \frac{1}{d}\left[\left(-\frac{e^{-jkd\sin\theta}}{jk\sin\theta}\right) - \left(-\frac{1}{jk\sin\theta}\right)\right]. \tag{23.7}$$

Now lets take the magnitude of this directivity factor. First, we simplify our top equation a little,

$$D(\theta) = \frac{1 - e^{-jkd\sin\theta}}{jkd\sin\theta} \tag{23.8}$$

before employing Euler's formula to replace the complex exponential

$$D(\theta) = \frac{1 - \cos(kd\sin\theta) + j\sin(kd\sin\theta)}{jkd\sin\theta}. \tag{23.9}$$

Now taking the magnitude we end up with,

$$D(\theta) = \frac{\sqrt{(1 - \cos(kd\sin\theta))^2 + (\sin(kd\sin\theta))^2}}{kd\sin\theta}. \tag{23.10}$$

Note that the denominator is outside of the square root as it's a shared term between the real and imaginary parts, so when its squared we can bring it outside the square root.

To make life a little easier lets just substitute $x = kd\sin\theta$. We can put it back in later. Now lets expand the $i - \cos x$ term.

$$D(\theta) = \frac{\sqrt{1 - 2\cos(x) + \cos^2(x) + \sin^2(x)}}{x}. \tag{23.11}$$

Notice that we get a $\cos^2(x) + \sin^2(x)$. This is just equal to 1.

$$D(\theta) = \frac{\sqrt{2 - 2\cos(x)}}{x}. \tag{23.12}$$

Now substituting $x = kd\sin\theta$ back in and factoring out the constant 2,

$$D(\theta) = \frac{\sqrt{2(1 - \cos(kd\sin\theta))}}{kd\sin\theta} \tag{23.13}$$

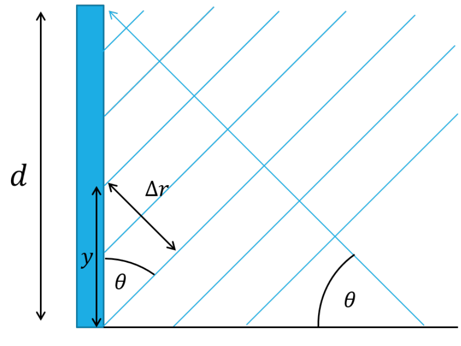


Figure 23.2: Geometrical set up for phase shift directivity

we get to a point where we can use another trig identity; $1-\cos(x) = 2\sin^2(x/2)$.

$$D(\theta) = \frac{\sqrt{2^2 \sin^2(\frac{kd\sin\theta}{2}))}}{kd\sin\theta} \tag{23.14}$$

Now we have the square root of some squares, so we can simplify that, and voila,

$$D(\theta) = \frac{2\sin(\frac{kd\sin\theta}{2}))}{kd\sin\theta} \tag{23.15}$$

we arrive at our directivity function for a diaphragm due to a phase shift along its length. This is assuming a 1d diaphragm by the way. Although a 1d diaphragm might not be very useful in reality, it does a good job telling us whats going on.

So what does this directivity look like? A few example polar responses are shown in figure 23.3 for different diaphragm sizes. This directivity due to a phase change effects all diaphragms. The effect of this is one of beaming; diaphragms of a finite size are more directional at high frequencies.

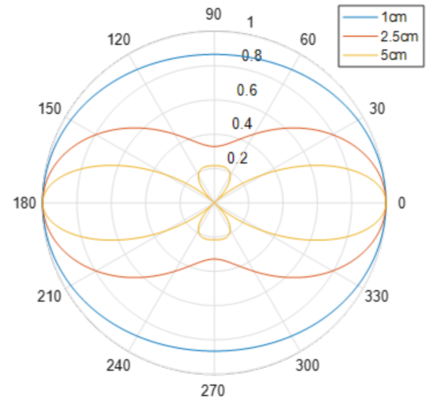


Figure 23.3: Directivity response due to phase shift over diaphragm.

This makes sense right? On axis there is no change of phase so this effect doesn't happen. Off axis, the response depends on frequency and diaphragm size, together these dictate the number of phase changes we get over the length of the diaphragm.

Remember, we are assuming this things moves as a piston. So the pressure over the surface is averaged to a single forcing term which moves the diaphragm as a single lump. When there are lots of phase changes the integrated force may well be zero(!) simply due to the finite size of the diaphragm. In this case there is no resultant displacement.

23.2 Diffraction

Now that we have covered the first source of directivity, due to phase cancellations over a diaphragm, we can consider its second source. This second source of directional behaviour arises due to the fact that the microphone disturbs the sound field it is trying to measure.

The impedance of a microphone diaphragm is generally high compared to free space. This means that the presence of a diaphragm changes the sound field (it introduces reflections). In fact, for microphones to work they have to disturb the sound field. Something needs to be moved, some energy must be removed from the wave and converted into an electrical signal.

When a microphone disturbs a sound field there are two processes going on: reflection, and diffraction. Whilst some of the incident wave is reflected of the diaphragm, some is diffracted *around* the diaphragm. The net force on the diaphragm is related to the total pressure on its surface. Now we have to consider the diffracted contribution, as this changes the total pressure on the surface, i.e. some sound is diffracted around the microphone. But this means that whatever is behind the microphone also has an effect! So not only the diaphragm, but the casing also. Turns out that the shape of the microphone casing can be the cause of some unexpected lumps in the frequency response.

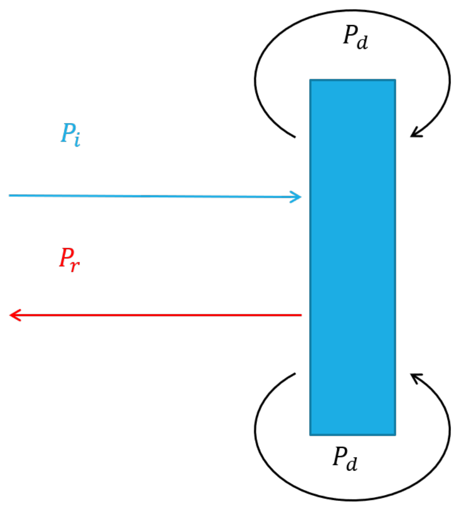


Figure 23.4: Reflection and diffraction around microphone diaphragm.

The smaller the diaphragm (and microphone) the less the microphone changes sound field. So why not make microphones really small? Well, the smaller the diaphragm the poorer the signal to noise ratio, as the force that's transduced is effectively reduced.

So it turns out that there is no such thing as a perfectly omnidirectional microphone. This would require an infinitely small device. But you can make them pretty small. The wavelength of sound at 20 kHz is about 17 mm. You probably want your diaphragm to be an order of $1/8$th of that to have no effect on sound field. But that includes the casing. Clearly there are some serious design challenges in achieving this.

Now lets think about the extreme limits of diffraction and reflection. At very very high frequencies, were diffraction is negligible, the total pressure on the diaphragm is that of the incident plus reflective. We can approximate this as simply 2 times the incident pressure.

$$p_T = p_i + p_r \overset{\omega\to\infty}{\approx} 2P_i \tag{23.16}$$

At very low frequencies, were the microphone isn't large enough to cause a significant reflection, what was reflected is now diffracted around the microphone. All we have over the diaphragm is the incident pressure.

$$p_T = p_i + p_r \overset{\omega\to 0}{\approx} P_i \tag{23.17}$$

From these extreme limits it is clear that diffraction has the effect of introducing a pressure doubling at high frequencies (i.e. a 6 dB boost).

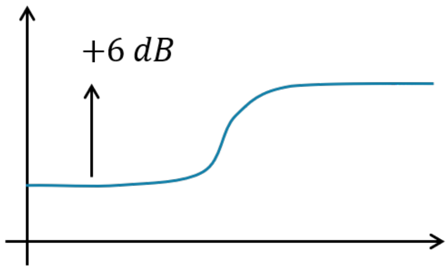


Figure 23.5: Pressure doubling at high frequencies due to reflections.

Lets look at modelling the effect of diffraction around a microphone diaphragm. For mathematical convenience we will treat our diaphragm as an infinite strip (into the screen), but with a finite width (d) which is equivalent to the diaphragm width. The total pressure is given by the incident plus the reflected,

$$p_T = p_i + p_r. \tag{23.18}$$

The amount of diffraction around a strip is given by this equation,

$$p_d = \frac{p_i \sin(kd\sin\theta)}{kd\sin\theta} \tag{23.19}$$

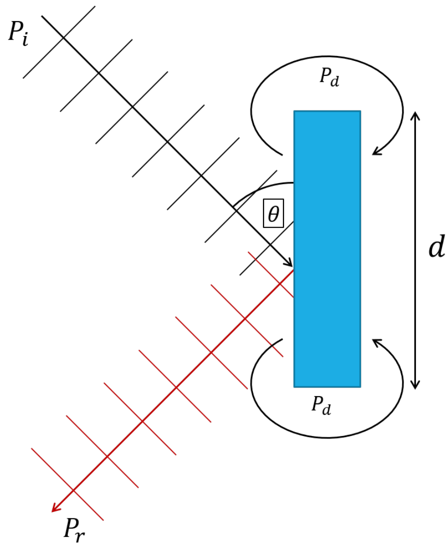


Figure 23.6: Incident, reflected and diffracted pressure on a diaphragm.

We wont derive this equation here, we'll just take it as gospel. Assuming a rigid diaphragm, which strictly isn't true but is a good approximation here, the reflected pressure is the incident minus the diffracted, i.e. it is what remains to be reflected after some of the incoming wave may have been diffracted around,

$$p_r = p_i - p_d. \tag{23.20}$$

Combining equations 23.18, 23.19 and 23.20 together gives us the total pressure on the diaphragm including the effect of diffraction.

$$p_T = p_i\left(2 - \frac{p_i \sin(kd\sin\theta)}{kd\sin\theta}\right). \tag{23.21}$$

Noting that $\sin(x)/x = 1$ for small values of x, this equation provides the interpolation between our two limit cases of: the incident pressure only, and a pressure doubling. Clearly it is angle dependent, but taking the on axis response $\theta = 0$ as an example, at high frequencies the since function tends to zero. At very low frequencies, it tends to one. As expected.

So what effect does this have on a microphones response? Shown in figure 23.7 is the on-axis response for a 5 cm diaphragm. The orange curve represents our diaphragms displacement response in the absence of diffraction/reflection. In blue is the total pressure response including the effect of diffraction/reflection, but neglecting the mechanical model. In yellow is the dynamic model including the effect of diffraction/reflection. It is clear that we get a significant boot at high frequencies.

This response poses an interesting question; do we want this $+6$dB boost? What can we do to make this work better? We can change the resonant frequency so that the mechanical roll off occurs before this boost. By making the diaphragm less stiff we can move the resonance down so that it rolls off before we get into the $+6$dB boost. So you can account for this boost by controlling the diaphragm dynamics. This in figure 23.8 is the resulting response. Notice that we get a much flatter response now. Also notice, that the reflection effect extends our frequency response slightly.

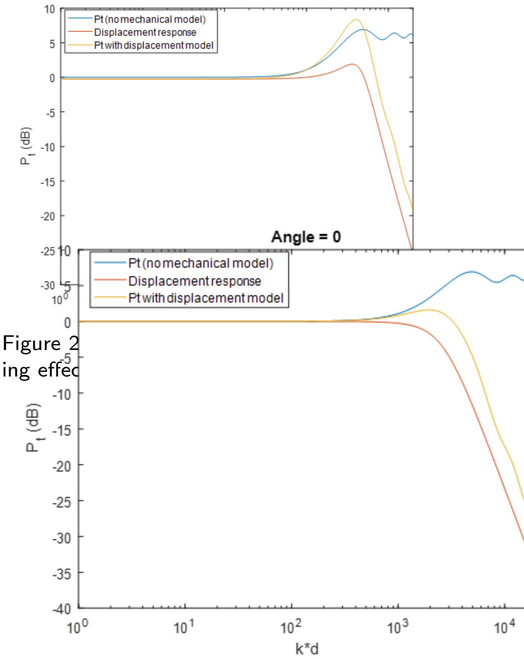


Figure 2
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Figure 23.8: On axis frequency response including effect of reflection and diffraction with a reduced resonant frequency.

So it should be clear by now that you will never get a perfectly flat response, especially with large diaphragms. It turns out that many a microphone has been built, perhaps accidentally, to give a lumpy response due to some of these effects. These lumps have turned out to be pleasant to the ears so people have kept them.

Now suppose we reduce the stiffness to correct for this high frequency boost, thus giving us a flat frequency response on axis. What are the consequences off-axis? Moving off-axis has the effect of pushing up (in frequency) the point where the 6 dB boost kicks in. Take the extreme limit at $90°$ for example; we get no reflections, so the response is always 1 (the frequency of the boost has been pushed all the way up to infinity).

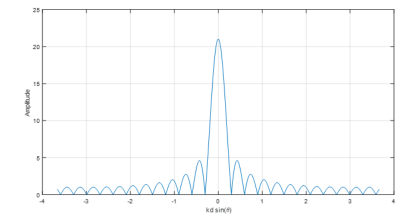


Figure 23.9: Sinc function.

This means that while the response is equalised on axis, the off axis response is not flat. The high frequencies have been rolled off more than we may desire (because we are no longer benefiting from the reflection boost). See for example figure 23.10 which shows the frequency response at 3 angles: $0°$, $70°$ and $90°$.

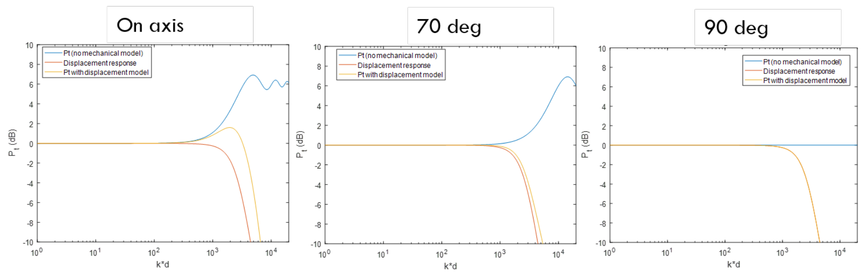


Figure 23.10: Angular dependence of frequency response on effect of reflection and diffraction with a reduced resonant frequency.

Shown in figure 23.11 and 23.12 are some examples of the polar patterns of two different sized diaphragms. In both cases we can see that as frequency is increased, the microphone gets more directional. Also, because of the $sin(x)/x$ function in the diffraction term we get this odd shape at higher frequencies (can be greater than 6 dB boost due to ripples in $sin(x)/x$ function).

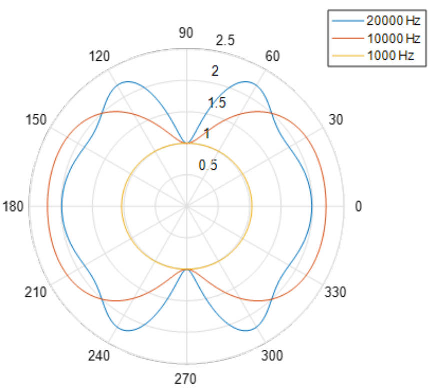
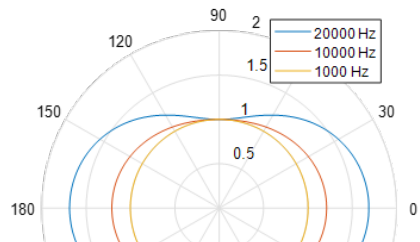


Figure 23.11: Polar pattern of 1 inch diaphragm.

Comparing the 1 inch and $1/4$ inch diaphragms, the $1/4$ inch microphone has more diffraction going on with less reflections, hence has more even response (less reflected pressure means less of a boost). For the large diaphragm the sound is more likely to be reflected (less diffraction), so the boost occurs at lower frequencies.



24 Microphone Characteristics

24.1 Directivity Measures

We have talked a little bit about the unavoidable causes of directivity (phase changes and diffraction), now lets talk about directivity a little more generally.

So far we have seen two types of directivity. When considering monopole sources and pressure sensing microphones we had the omni-directional response.

$$p_T = A_0 e^{j\omega t} \tag{24.1}$$

When considering dipole sources and pressure gradient sensing microphones we had the figure of 8 response.

$$p_T = A_0 e^{j\omega t} jkd \cos\theta \tag{24.2}$$

In fact, when looking at delayed dipole sources we also saw the cardioid response.

$$p_T = A_0 e^{j\omega t} jkd(1 + \cos\theta) \tag{24.3}$$

These are just three of an infinite number of possible directivity patterns.

Directivity patterns can be described in terms of their order. High order directivities have more detail than lower ones. A zeroth order directivity is simply an omni-directional monopole. A first order directivity, corresponds to a linear combination of an omni-directional response, with a figure of 8 response, i.e. for example if you were to combine a pressure sensing component (omni-directional) to a pressure gradient (figure of 8) sensing component, as in figure 24.1.

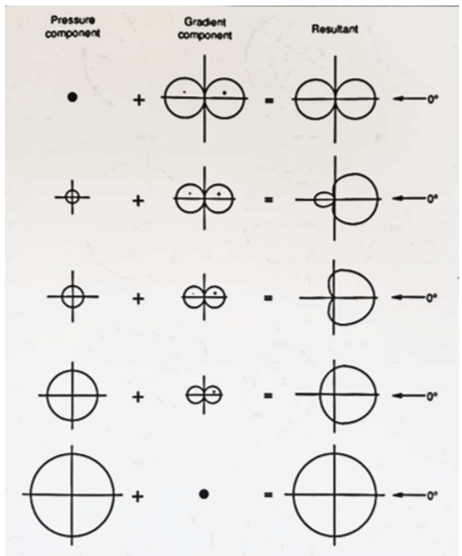


Figure 24.1: Examples of first order directivities.

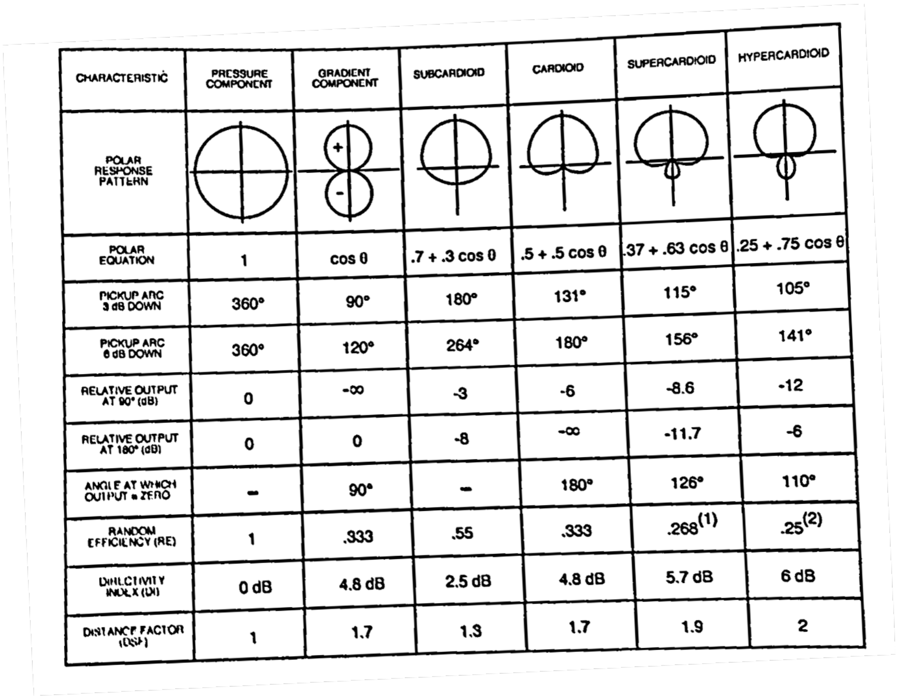
The general form of a first order directivity pattern is given by,

$$g = (1 - B) + B\cos\theta. \tag{24.4}$$

It has the constant (pressure sensing) term $(1 - B)$ and the angular dependent (pressure gradient sensing) $B\cos\theta$. The value of B determines which pattern we get. For example with $B = 0$, there is no $\cos$ term so it is omni-directional. If $B = 1$, there is no omni term so it is all $\cos$, i.e. a figure of 8. If we have half of each $B = 0.5$ we get the cardioid pattern.

There are loads more of patterns in between these ones. Two common ones are; $B = 0.3$ – the sub-cardioid response (slightly directional response) and $B = 0.75$ - the hyper-cardioid response (very directional response, with that trade off that you get a small response contribution from behind).

Shown in figure 24.2 is a table with some useful characteristics of different directivity patterns. One particularly interesting feature is the right angle rejection. This is a measure of the relative level at $90°$. From figure 24.2 we see that it can



CHARACTERISTIC	PRESSURE COMPONENT	GRADIENT COMPONENT	SUBCARDIOID	CARDIOID	SUPERCARDIOID	HYPERCARDIOID
POLAR RESPONSE PATTERN						
POLAR EQUATION	1	cos θ	.7 + .3 cos θ	.5 + .5 cos θ	.37 + .63 cos θ	.25 + .75 cos θ
PICKUP ARC 3 dB DOWN	360°	90°	180°	131°	115°	105°
PICKUP ARC 6 dB DOWN	360°	120°	264°	180°	156°	141°
RELATIVE OUTPUT AT 90° (dB)	0	-∞	-3	-6	-8.6	-12
RELATIVE OUTPUT AT 180° (dB)	0	0	-8	-∞	-11.7	-6
ANGLE AT WHICH OUTPUT = ZERO	–	90°	–	180°	126°	110°
RANDOM EFFICIENCY (RE)	1	.333	.55	.333	.268(1)	.25(2)
DIRECTIVITY INDEX (DI)	0 dB	4.8 dB	2.5 dB	4.8 dB	5.7 dB	6 dB
DISTANCE FACTOR (DSF)	1	1.7	1.3	1.7	1.9	2

Figure 24.2: Table of directivity information.

vary from a measly 3dB (sub-cardioid) to 12 dB (hyper-cardioid). This is quite a large difference in directivity.

Another interesting characteristic is the angle at which the output is zero. This is the angle where you get a null is the directivity. For cardioid it is directly behind the microphone ($180°$). In a live sound situation you want this microphone pointing directly away from stage monitors, to prevent feedback. For the super-cardioid the zero output angle is $126°$. So you would position this microphone at a slightly different angle to the wedge monitors. The zero for a hyper-cardioid response is different still, requiring a near horizontal alignment to any monitor wedges.

The last three characteristics in figure 24.2, the random efficiency RE, directivity index DI, and distance factor DF, each provide a single value measure of a microphones directivity, relative to an omni-directional response. We will briefly discuss each of these below.

24.1.1 Random Efficiency

The voltage output of a microphone is related to the incident pressure P (which can be angle dependent), weighted by the microphones frequency dependence K (sensitivity and transduction), and also weighted by the angular dependence of the microphone itself, g.

$$v(f,\theta) = p(\theta)K(f)g(\theta) \tag{24.5}$$

Note that K and g are just amplitude weighting factors, so they are purely real.

Assuming a incident plane wave, the effective intensity is approximately,

$$I = \frac{|p|^2}{\rho_0 c} \propto \frac{|v|}{\rho_0 c} \tag{24.6}$$

Now remember, power is intensity times area. Using this we can work out the total acoustic power of the microphone,

$$W = IA \tag{24.7}$$

To determine the total acoustical power of a microphone we need to consider sound coming in from all directions, i.e. we need to integrate the intensity over the area of a sphere.

$$W = \int IdA \tag{24.8}$$

We do this by examining an imaginary sphere that surrounds the microphone. We can make the reasonable assumption that the microphones directivity has rotational symmetry. Now to work out what gets captured we can work out the incident power over a series of thin strips around the sphere. Then we add up (integrate) the power over all strips sphere to get the total power.

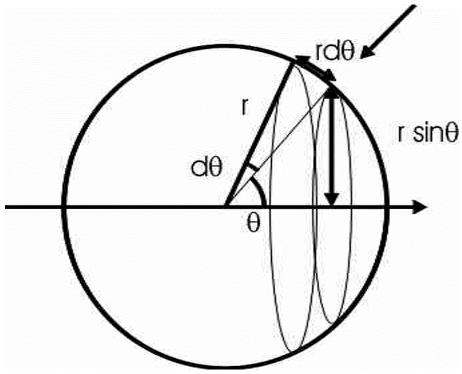


Figure 24.3: Integration over a sphere.

The strip wraps around the sphere to form a circle. The radius of each circle is $r\sin\theta$, its circumference is $2\pi \times r\sin\theta$. To think about strip width we move an infinitesimally small angle away from theta, $d\theta$. The length of the arc, is then $rd\theta$. Therefore the area of each strip as it is wrapped around the sphere is this arc length $rd\theta$ times the circumference of the strip, i.e. $rd\theta \times 2\pi r\sin\theta$.

The total incident power is the surface integral of the intensity over the sphere. This is quite hard to work out. We can instead replace dA with what we worked out for the infinitesimal area assuming rotational symmetry. Also introducing our expression for intensity $I \propto |v|/\rho_0 c$, we have a normal integral with respect to angle θ,

$$W \propto \int_0^\pi \frac{|v|}{\rho_0 c} rd\theta \times 2\pi r\sin\theta = \frac{2\pi r^2 K^2(f)}{\rho_0 c} \int_0^\pi |P(\theta)|^2 g^2(\theta)\sin\theta d\theta. \tag{24.9}$$

Next we make the assumption of a diffuse field. This means that sound waves arrive from all directions equally (on average). This gets rid of the pressure angle dependence,

$$W \propto \frac{2\pi r^2 |p|^2 K^2(f)}{\rho_0 c} \int_0^\pi g^2(\theta)\sin\theta d\theta. \tag{24.10}$$

Voila, we have an expression that is proportional to the total output power of the microphone. What will this be for an omni-directional microphone? The directivity factor $g^2(\theta) = 1$, so we get a factor of 2 since $\int_0^\pi = \sin\theta d\theta = 2$.

Now that we have an expression for the output power of a microphone, we can define the Random Efficiency (RE),

$$RE = \frac{W_{diffuse,directional}}{W_{diffuse,omni}}. \tag{24.11}$$

The RE is the ratio of the power output of the microphone in a diffuse field to the power output of an omni microphone with the same sensitivity (K), in the same diffuse field.

Substituting in the expression for power we get,

$$RE = \frac{\frac{2\pi r^2|P|^2K^2(f)}{\rho_0 c}\int_0^\pi g^2(\theta)\sin\theta d\theta\times}{\frac{2\pi r^2|P|^2K^2(f)}{\rho_0 c}\int_0^\pi \sin\theta d\theta} = \frac{\int_0^\pi g^2(\theta)\sin\theta d\theta\times}{\int_0^\pi \sin\theta d\theta} = \frac{\int_0^\pi g^2(\theta)\sin\theta d\theta\times}{2}. \tag{24.12}$$

We are left with a straight forward expression, we just need to integrate the polar pattern with a $\sin$ term. This gives us a number that describes how directional the microphone is. Essentially, the lower the number the less energy the directional microphone grabs from the diffuse field relative to an omni-directional microphone. So a smaller number gives more directional response.

24.1.2 Directivity Index

The Random Efficiency as derived above is related to another common measure of directivity, the so called directivity factor.

$$D = \frac{1}{RE}. \tag{24.13}$$

The directivity factor is effectively the same parameter, though usually taken as the reciprocal of the random efficiency. The idea being that the bigger the number the more directional the microphone. If we take the log of this and multiply by ten we get another measure, the directivity index,

$$DI = 10 \log_1 0(D). \tag{24.14}$$

The directivity index is a simply a logarithmic measure of directivity.

24.1.3 Distance Factor

Another useful parameter is the distance factor. This is a measure of the 'reach' of the microphone. For example, a microphone with a distance factor of 2 means, in a reverberant environment it can be placed at twice the distance from the source compared with an omni-directional microphone, to achieve the same direct to reverberant ratio. For example, you can place a hyper-cardioid microphone twice as far away from source and get the same(ish) sound as an omni directional microphone.

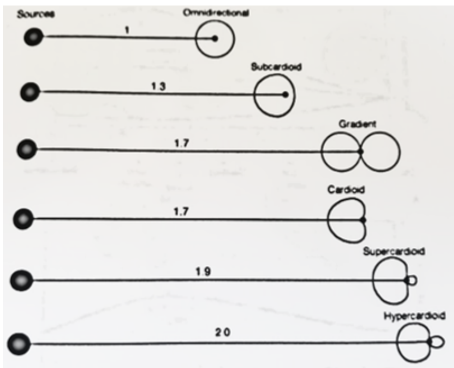


Figure 24.4: Distance factors for common first order directivites.

How do we calculate this? We use the expression for the random efficiency. Before we cancelled out the distance r, because we were considering the same distance for both microphones. Now we allow the distance of the directional and omni microphones to be different: r_d is the distance of the source from the directional mic, and r_0 is the distance of the source from the omni microphone.

$$\frac{r_d^2 \int_0^\pi g^2(\theta) \sin\theta d\theta \times}{r_0 \int_0^\pi \sin\theta d\theta} \tag{24.15}$$

The question is then 'where do we put the directional microphone in relation to the omni-directional microphone, so that they have the same overall power from the source?' This is the same as setting the ratio equal to one, and finding the ratio of distances,

$$\frac{r_d^2 \int_0^\pi g^2(\theta) \sin\theta d\theta \times}{r_0 \int_0^\pi \sin\theta d\theta} = 1. \tag{24.16}$$

What we end up with is the square root of the reciprocal of the random efficiency,

$$1 = \frac{r_d^2}{r_0^2} RE \rightarrow DF = \frac{r_d}{r_0} = \frac{1}{\sqrt{RE}}. \tag{24.17}$$

24.2 Diffuse vs. Free Field

All this directivity talk is hinting towards an important point. If we equalise the microphone according to the forward direction, we get a great response for free-field measurements. But when the microphone is in a reverberant environment we

get a reduced high frequency response, since the off-axis response is attenuated at high frequencies.

Whilst in some situations, e.g close micing of an instrument, the on-axis response was what's most important, we often want to measure pressures in more reverberant fields (for example when measuring absorption coefficients), where the off-axis response is dominant. If you used a 'free field' calibrated microphone you would loose some high frequency energy. So what's the solution?

Figure 24.5: Diffuse field measurement, or close micing.

We want a microphone that gives a flat response in a diffuse environment, so lets equalise its response to achieve exactly this. By doing so we ensure that a diffuse sound field is measured with a flat frequency response, but of course the microphone is a little bit more directional in one direction, and so would be very bright if use to close mic something. The main point is that the microphones are the same they are just equalised differently. This is why there are different types of microphone.

- The free field microphone is designed so on axis its response is flat. Hence this tries to accounts for it's own presence in the sound field so forward facing is flat.

- The pressure microphone doesn't try to equalise out it's own response. It accepts that there may be some reflections. This is useful when you don't want these equalised out, such as when the microphone is mounted flush to a wall.

- The random incidence microphone is designed to give a flat response when averaged over sound arriving from all directions.

All these microphones can be used to measure a free field. The free-field mic would be pointed towards to the source. The pressure mic, which has a flat response when there are no reflections, is placed at 90 degrees to the source (although this does compromise the sensitivity). The diffuse field microphone is usually can be placed between 70-80 degrees to the field to achieve a flat frequency response.

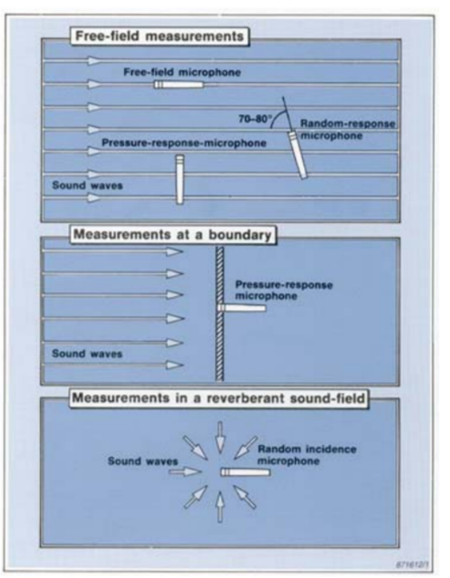


Figure 24.6: Free-field, boundary and reverberant field measurements.

24.3 Sensitivity

Another important microphone characteristic is its sensitivity. The sensitivity of a microphone is defined as the open circuit voltage due to a 1 Pa pressure: v/Pa, mv/Pa, or in dB re $1v/Pa$.

When looking at loudspeaker design (actually cross-over design) we saw that in order for an amplifier to deliver the same voltage whatever the load, it needed to have a very very low impedance. This means that it functions like an ideal voltage source. We need to consider a similar thing when we look at the sensitivity of microphones, to make sure what we measure is dependent only on the microphone.

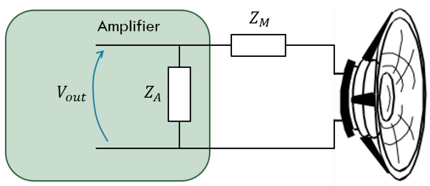


Figure 24.7: Measurement of microphone sensitivity.

To do this we measure the microphone using a device with a very high impedance. This ensures that only the microphone influences the measurement. For example, consider the set up in figure 24.7, a microphone with internal impedance Z_M is coupled to an amplifier with internal impedance Z_A. The output voltage is given by,

$$v_{out} = Z_T i. \tag{24.18}$$

The total impedance is that of the amplifier in parallel with the microphone,

$$Z_T = \frac{Z_A Z_M}{Z_A + Z_M}. \tag{24.19}$$

Multiplying top and bottom by $1/Z_A$ yields,

$$Z_T = \frac{Z_M}{1 + \frac{Z_M}{Z_A}}. \tag{24.20}$$

The above equation quite clearly shows that to measure the properties of the microphone independently (i.e. without any effect of the amplifier), we need Z_A to be as large as possible,

$$Z_T \xleftarrow{asZ_A \to \infty} Z_M. \tag{24.21}$$

24.4 Distortion

Another really important characteristic of a microphone that is commonly stated in specification sheets is that of the Total harmonic Distortion (THD). We have been assuming our microphone systems are nice and linear. In reality, all microphones exhibit some amount of non-linearity. What does this mean? Simply put, if the microphone were subject to an absolute pure tone, the output voltage would be this pure tone, plus a series of 'extra tones'. These extra tones (also called over-tones) arises due to non-linearity in the system. The underlying causes of non-linearity are complex, and beyond the scope of this model. Some typical examples include a displacement dependent force factor $B(x)L$, or a displacement dependent suspension stiffness $k(x)$.

The total harmonic distortion is a measure that quantifies the non-linearity of a device (can also be used for loudspeakers). It is the ratio of the rms of all the overtones except the fundamental, to the rms level of the fundamental. It is often expressed as a percentage.

$$THD = \frac{\sqrt{v_2^2 + v_3^2 + v_4^2 + \cdots}}{v_1} \times 100 \tag{24.22}$$

The THD can be used to express what the maximum useful sound pressure level is for a microphone. For example, you may see something like, *Max SPL* 139 *dB (*1 *kHz, ¡* 1*%* THD*)*. This is saying the THD is less than 1% when the SPL is less than 139 dB. It is interesting to note that the THD is a frequency dependant quantity. Usually the value at 1 kHz quoted, but you could run a frequency sweep to get the THD at every frequency.

Another important point; it turns out that the THD is actually not a very good indicator of the perceptual quality of something. People tend to like odd harmonics. That is why we don't use it to assess perceptual levels of distortion. For this we use criterion such as PEAQ (Perceptual Evaluation of Audio Quality), PESQ (Perceptual Evaluation of Speech Quality), etc. These have auditory models built in.

24.5 Noise

Another important consideration in microphone (and amplifier) design is noise. Microphones do not provide much of an output. We are talking 0.5 mV to 22 mV response to 1 pascal (remember 1 Pa (1 Newton /m2) is $+94$dB SPL).

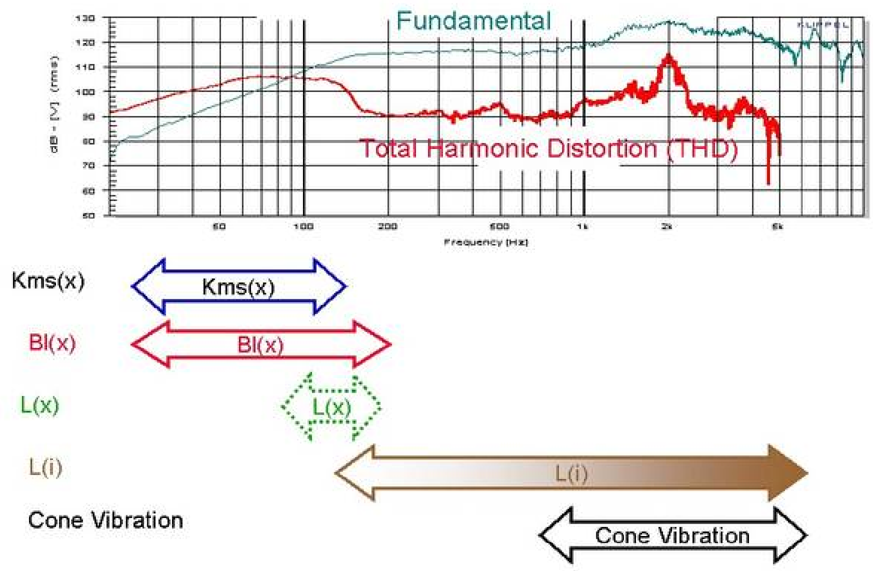


Figure 24.8: Total Harmonic Distortion

Considering that a normal voice level at 1 m is 60-70 dB, this is a very low output. Its is approx. 20 dB lower than 94 dB, which is a factor of 10. So a microphone output would likely be in the range of 0.05mV to 2.2 mV. So what do we do? We use amplifiers to boost the signal.

Amplifiers don't just boost the signal we want though, they boost everything that comes with it. I.e. if we have any noise in our system, that also gets amplified. Another thing to remember, amplifiers introduce their own noise also. So for microphone design, noise is definitely an issue.

To boost a signal we often use multiple amplifiers. Rather than trying to boost the signal in one big go, we do it bit by bit. This tends to reduce problems such as non-linearity. When using staged amplifiers like this, the first amplifier is critical. Any noise introduced by this amplifier will get multiplied all the way down the line.

What are the main sources of noise in microphone systems? There are three: acoustic noise, thermal noise, and shot noise.

Acoustic noise is pretty obvious. This is background noise in the recording environment that you didn't intend to record. This sort of noise can be reduced for example by controlling the directivity of the microphone (perhaps to reduce sounds coming from behind), or maybe making use of the proximity effect.

24.5.1 Thermal Noise

Thermal noise (also called Johnson-Nyquist noise) is due to the random thermal movement of electrons inside an electrical conductor. Thermal noise is similar to Brownian motion. It occurs in any conductors, whether there is a current there or not. It is present in all electrical circuits and is a function of temperature and resistance.

Since thermal noise is a stochastic process, there is no deterministic model (no defined waveform) for its generation. As such we can only look at its power spectral density (PSD), and other related statistics. The average squared voltage level due to thermal noise is given by,

$$\mathbb{E}[v^2] = 4KTRB \tag{24.23}$$

where K is Boltzman's constant, $T = 1.38\times10^{-23}$ is the temperature (in Kelvin), R is resistance, and B is the frequency bandwidth of interest. Its RMS level is then given by,

$$\sqrt{\mathbb{E}[v^2]} = \sqrt{4KTRB}. \tag{24.24}$$

In an ideal resistor thermal noise is approx. white (i.e. constant PSD across all frequencies) and when limited to a finite bandwidth has an approx. Gaussian amplitude distribution.

The effect of thermal noise can be reduced by lowering the temperature. This is often done when conducting experiments with very sensitive equipment.

24.5.2 Shot Noise

Shot noise is entirely different to thermal noise, and arising due to the discrete nature of electric charge causing random fluctuations of the electrical current (i.e. current is not a continuous flow, it is a sum of discrete pulses in time, each corresponding to the transfer of an electron through the conductor.) Acoustically, shot noise sounds quite similar to rain on a tin roof.

Like thermal noise, shot noise is independent of frequency, but unlike thermal noise, it is independent of temperature. Instead shot noise depends on the level of current. Its averaged squared level is given by,

$$\mathbb{E}[I^2] = 2eIB \tag{24.25}$$

where $e = 1.602 \times 10^{-19}$ is the charge of an electron, I is the current, and B is the frequency bandwidth of interest.

Shot noise occurs in all electrical components but is mainly generated within the amplifiers, for instance in the transistor junctions. It is usually orders of magnitude lower than thermal noise, but in low temperature applications it can be a problem.

25 Designing for a Directional Response

So far we have talked a little bit about directional microphone responses. In particular we have seen the omni-directional pressure sensing microphone, and the figure of 8 pressure gradient sensing microphone. We know that there exists a whole load of other useful directivity patterns; how can we design a microphone to achieve one of these other directivity patterns?

25.1 Phase Shift Method

One approach would be to incorporate some sort of phase shift between the front and rear of a diaphragm. This is essentially what we did with the cardioid subwoofer design. In a microphone context we could do this using a tube, and placing the diaphragm at one end. Remember, if you place the diaphragm in the middle, you get a figure of 8 pattern. Place it at one end and the additional delay is equivalent to the length of the tube. So we have a phase delay due to angle ($d\cos\theta$), and due to the length of the tube (d).

We wont go through it here, but I expect you to be able to go from from p_f and p_b,

$$p_f = p_0 e^{j(\omega t - kr)} \tag{25.1}$$

$$p_b = p_0 e^{j(\omega t - k(r + d + d\cos\theta)} \tag{25.2}$$

to the expression for total pressure,

$$p_T = jkdp_f(1 + \cos\theta) \tag{25.3}$$

The directivity factor you end up with is that of a cardioid pattern. The maths is exactly the same as the cardioid subwoofer.

For this method to work then we need to a small tube, so that no resonant behaviour occurs in the tube. The problem is that the smaller the spacing the smaller the response due to the $jkdp_f$ term. So there is a trade off. Main point, it is difficult to use a tube to design for directivity.

Here is another idea, why don't we use two diaphragms with an electronic tune-able delay, like we did with the sub-woofers? We will go through this idea later. For now we will stick with the most common approach, using acoustic circuits.

25.2 Acoustic Filtering Network - Helmholtz

Shown in figure 25.2 is a diagram of a typical capacitor microphone. Note that there is a hole at the rear, this is very important.

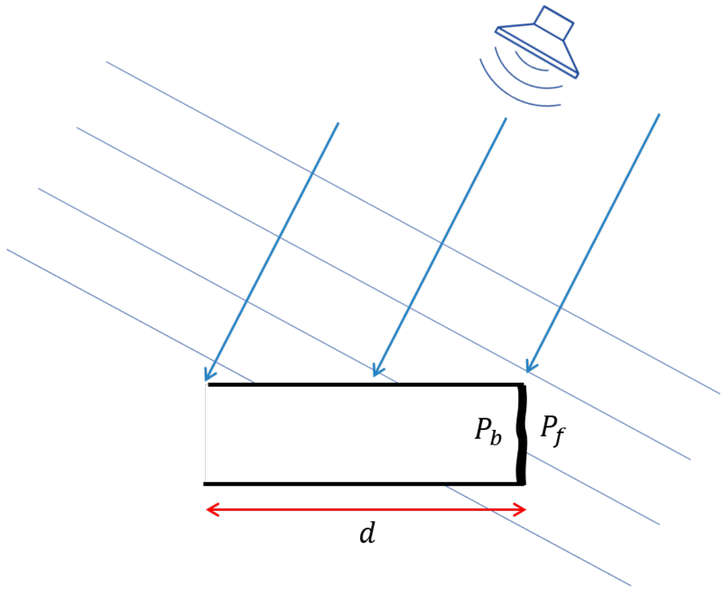


Figure 25.1: Phase shift design method for directional microphone response.

The pressure at the front is p_f and the pressure at the rear is p_b. We can represent this system as an acoustic circuit.

We have two pressure generators corresponding to p_f and p_b. We have a diaphragm (whose dynamics can be represented by a mass-spring-damper system, which itself can be represented by an LCR series circuit). We have a hole at the rear which (assuming the plug of air moves as one) we can model as an acoustic mass/resistance (i.e. an inductor / resistor). We also have a cavity, which we can model as a compliant volume (i.e. a capacitor), maybe with some losses (i.e. another resistor).

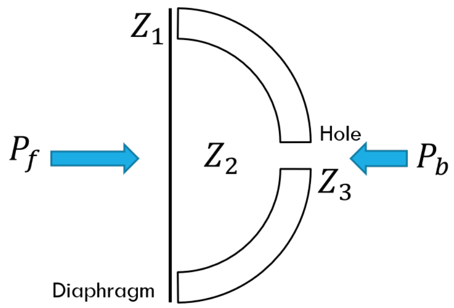


Figure 25.2: Diagram of conventional capacitor microphone design.

Shown in figure 25.3 is the equivalent circuit. We have a potential divider, from the front and the back. But there is are two active components (pressure generators) hidden away. This makes the analysis is a bit more complicated than a simple potential divider.

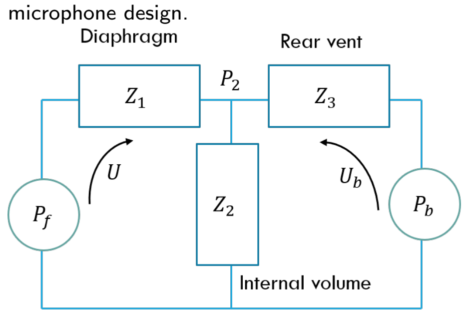


Figure 25.3: Equivalent circuit for a conventional capacitor microphone design.

Using this equivalent circuit, we are interested in determining the microphones sensitivity. To do this we have to determine the voltage output given a 1 Pa input pressure. Remember, the output voltage of a capacitor microphone is displacement sensitive. Our first step will be to determine the displacement of the diaphragm. This will mean solving the equivalent circuit for the diaphragm volume velocity, and then converting to displacement. Now lets derive an expression for the velocity of the diaphragm.

Using Ohm's law we can describe the potential drop over the each impedance in figure 25.3. The potential drop over Z_1,

$$p_f - p_2 = UZ_1. \tag{25.4}$$

The potential drop over Z_3,

$$p_b - p_2 = U_b Z_3. \tag{25.5}$$

The potential drop over Z_2,

$$p_2 = (U + U_b)Z_1. \tag{25.6}$$

From these three equations we can derive expressions for the front and rear pressures. The front pressure is given by,

$$p_f = UZ_1 + p_2 = UZ_1 + (U + U_b)Z_2 \tag{25.7}$$

which simplifies to,

$$p_f = U(Z_1 + Z_2) + U_b Z_2. \tag{25.8}$$

The rear pressure is given by,

$$p_b = U_b Z_3 + p_2 = U_b Z_3 + (U + U_b)Z_2 \tag{25.9}$$

which simplifies to,

$$p_b = U_b(Z_3 + Z_2) + UZ_2. \tag{25.10}$$

We now have a pair of simultaneous equations for the front/rear pressure, and the diaphragm/vent velocity. Since we are after the diaphragm velocity, lets solve for it. From equation 25.10 we get the rear vent velocity,

$$U_b = \frac{p_b - UZ_2}{Z_2 + Z_3}. \tag{25.11}$$

Substituting this into equation 25.8,

$$p_f = U(Z_1 + Z_2) + \frac{p_b - UZ_2}{Z_2 + Z_3} Z_2 \tag{25.12}$$

and simplifying yields,

$$p_f = \frac{U(Z_1Z_2 + Z_1Z_3 + Z_2Z_3) + p_b Z_2}{Z_2 + Z_3}. \tag{25.13}$$

Now we can rearrange to get the diaphragm volume velocity,

$$U = \frac{(Z_2 + Z_3)p_f - p_b Z_2}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3} \tag{25.14}$$

Now all we need is a useful expression for pressure at the rear vent. Luckily, we have done this kind of thing before. We know that the rear pressure will be related to the front pressure, but delayed by the extra distance the wave has to travel,

$$p_f = p_0 e^{j(\omega t - kr)} \tag{25.15}$$

$$p_b = p_0 e^{j(\omega t - k(r + d\cos\theta)} = p_f e^{-jkd\cos\theta}. \tag{25.16}$$

At low frequencies ($kd << 1$) we can simplify the rear pressure to,

$$p_b = p_f(1 - jkd\cos\theta). \tag{25.17}$$

Our diaphragm volume velocity is then given by,

$$U = \frac{(Z_2 + Z_3)p_f - p_f(1 - jkd\cos\theta)Z_2}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}. \tag{25.18}$$

Dividing both sides by the front pressure, and loosing a numerator Z_2, yields the acoustic transfer function of our microphone,

$$\frac{U}{p_f} = \frac{Z_3 + (jkd\cos\theta)Z_2}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}. \tag{25.19}$$

This transfer function describes the relation between the front pressure, and diaphragm volume velocity.

In deriving equation 25.19 we made no assumptions regarding the values of the impedance terms Z_1, Z_2, or Z_3. It turns out that by carefully choose these values, we are able to obtain different types of directivity pattern. Lets consider the cardioid pattern as an example.

The cardioid pattern is characterised by the directivity factor $1 + \cos\theta$. The question is then, what choice of Z_2 and Z_3 will give us a $1+\cos\theta$ in the numerator of our transfer function equation? Suppose, $Z_3 = jkdZ_2$, then,

$$\frac{U}{p_f} = \frac{jkdZ_2 + (jkd\cos\theta)Z_2}{Z_1Z_2 + Z_1jkdZ_2 + Z_2jkdZ_2} = \frac{jkdZ_2(1+\cos\theta)}{Z_1Z_2 + Z_1jkdZ_2 + Z_2jkdZ_2}. \quad (25.20)$$

From the above it is clear that to get a cardioid response, we need the rear vent impedance to equal $jkdZ_2$. How exactly do we achieve this?

Lets think about what Z_3 and Z_2 actually are. Z_3 is the mass and resistance of the vent,

$$Z_3 = j\omega M_{hole} + R_{hole} \quad (25.21)$$

and Z_2 is the cavity compliance,

$$Z_2 = \frac{1}{j\omega C_2}. \quad (25.22)$$

To get a cardioid response we need these to be equal. So lets equate them.

$$Z_3 = jkdZ_2 \rightarrow j\omega M_{hole} + R_{hole} = j\omega\frac{d}{c}\frac{1}{j\omega C_2} \quad (25.23)$$

Notice that the frequency dependence in the compliance term cancels,

$$j\omega M_{hole} + R_{hole} = \frac{d}{c}\frac{1}{C_2} \quad (25.24)$$

leaving an equation whose right hand side is a constant value. If the right hand side is constant, then the left hand side should also be constant. This requires the mass term to be 0,

$$Z_3 = R_{hole} = \frac{d}{c}\frac{1}{C_2}. \quad (25.25)$$

To get rid of the mass term we need to make the whole quite wide (e.g. a large opening, with a grid, or some porous material over it to add damping). This will make the damping the dominant term (remember acoustic mass is inversely proportional to surface area). The amount of damping required is determined by equation 25.20. It is related to the front-rear spacing d and the cavity compliance C_2.

Substituting equation 25.25 into our acoustic transfer function yields,

$$\frac{U}{p_f} = \frac{R_{hole}(1+\cos\theta)}{Z_1Z_2 + Z_1R_{hole} + Z_2R_{hole}}. \quad (25.26)$$

Also substituting in the cavity impedance,

$$\frac{U}{p_f} = \frac{R_{hole}(1+\cos\theta)}{Z_1\left(\frac{1}{j\omega C_2} + R_{hole}\right) + \frac{1}{j\omega C_2}R_{hole}} \quad (25.27)$$

rearranging slightly,

$$\frac{U}{p_f} = \frac{j\omega C_2R_{hole}(1+\cos\theta)}{Z_1\left(1 + j\omega C_2R_{hole}\right) + R_{hole}}. \quad (25.28)$$

and recalling that $R_{hole} = jkd1/j\omega C_2$,

$$\frac{U}{p_f} = \frac{jkd(1+\cos\theta)}{Z_1(1+jkd)+R_{hole}}. \tag{25.29}$$

Equation 25.29 represents the acoustic transfer function of our cardioid microphone. How did we get here? We made the rear vent purely resistive, this means a large opening, with a grid or some porous material over it to add damping.

We can use this exact same method to design microphones with other useful directivity patterns. The key equation is

$$\frac{U}{p_f} = \frac{Z_3 + jkdZ_2\cos\theta}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}. \tag{25.30}$$

The directivity depends entirely on our choice of Z_3. Recall the polar equation for first order directivity,

$$g = (1-B) + B\cos\theta. \tag{25.31}$$

Notice its similarity to the numerator of equation 25.30. The polar equation gives a cardioid response when by $B = 0.5$, i.e. when the constant term $1-B$ and the angular term B are equal. This is exactly how we obtained a cardioid response for our microphone, we set $R_{hole} = d/cC_2$. Lets think how we might design for other polar patterns.

Suppose the vent damping is given by some factor G times the cavity impedance, $R_{hole} = G \times d/cC_2$. For the cardioid response we have $G = 1$. For a figure of 8 response, there should be no damping (as if the rear were completely exposed), so $G = 0$. For an omni response there should be no rear contribution, so $G = \infty$. With reference to the polar equation it is clear that $G = (1-B)/B$. If $B = 0.5$ (cardioid) then $G = 0.5$, if $B = 1$ (figure of 8) then $G = 0$, and if $B = 0$ (cardioid) then $G = \infty$.

Now suppose we want to design for a sub-cardioid response where $B = 0.3$. This gives us $G = (1-0.3)/0.3 \approx 2.3$. To get a microphone with this directivity we set $R_{hole} = 2.3 \times d/cC_2$. Clearly all the other first order polar patterns are available this way too,

$$R_{hole} = \frac{1-B}{B}\frac{d}{c}\frac{1}{C_2}. \tag{25.32}$$

This is a really neat idea. By designing just the right rear vent such that its impedance is purely resistive, with just the right value, get any first order directivity we like. What's even more clever is that we can extend this idea to develop microphones with controllable directivities.

25.2.1 Variable Aperture

Shown in figure 25.4 is an early example of this idea, the RCA series 77 from 50s. This is a ribbon microphone. It has a cowl covering the rear side of the diaphragm. This cowl has a shutter where you can control the size of an aperture at the rear. This is a similar design to what we just worked through. The acoustic transfer function is given by the same equation as before,

$$\frac{U}{p_f} = \frac{Z_3 + jkdZ_2\cos\theta}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3} \tag{25.33}$$

with the difference that there is no compliant volume. Instead we have a resistive pipe or labyrinth. So now Z_2 is purely resistive rather than compliant,

$$Z_2 = R_2. \tag{25.34}$$

As before, Z_3 can have a mass and resistive part,

$$Z_3 = j\omega M_3 + R_3. \tag{25.35}$$

We can now use the same design approach as before. Lets consider the cardioid response to start with. A cardioid response is achieved when $Z_3 = jkdZ_2$. This corresponds to,

$$Z_3 = j\omega M_3 + R_3 = j\omega \frac{d}{c} R_2. \tag{25.36}$$

So to obtained a cardioid response we need to a) minimise the resistance R_3 (no losses in of rear aperture) and b) set the mass term to,

$$M_3 = \frac{d}{c} R_2. \tag{25.37}$$

The acoustic transfer function of this microphone is then given by,

$$\frac{U}{p_f} = \frac{Z_3 + jkdZ_2 \cos\theta}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3} = \frac{jkdR_2(1+\cos\theta)}{Z_1R_2 + Z_1jkdR_2 + jkdR_2^2} \tag{25.38}$$

which clearly has a cardioid directivity pattern.

The advantage of this variable aperture design is that the *aperture can be varied*. As we make the aperture smaller, its mass effect increases. This makes Z_3 the dominant term in the numerator, giving us a more sub-cardioid directivity. What about if we completely close the aperture? Well then we only sense the front pressure and we get an omni-directional response. What happens as we begin to open up the aperture? The mass term decreases, and so the $\cos\theta$ term becomes more and more dominant. At some point the aperture is of such a size that $M_3 = dR_2/c$ and we get exactly a cardioid response. As we open the aperture further the mass term tends to zero, and we end up with a figure of 8 directivity. These cases are all shown in figure 25.5. Such flexibility! All controlled by a variable aperture.

Figure 25.4: RCA 77D – Ribbon Microphone 1957

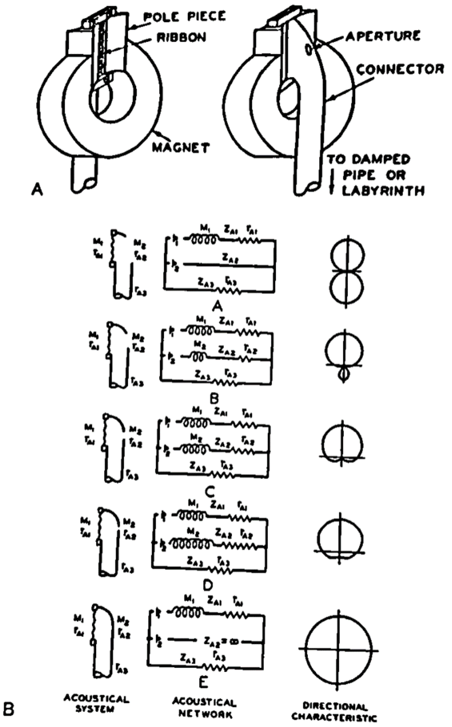


Figure 25.5: Diagram and equivalent circuits for different aperture openings.

25.2.2 Capsule Attachments

Here is another neat way to alter the directivity of a microphone; move the diaphragm. This is implemented on the AKG C-1000, very popular microphone. It's a really simple idea. We want to move the diaphragm from the end of a microphone, to some way down a tube, i.e. lengthen one side. For the C-1000 this is achieved using a adapter that is placed over the end of the microphone, as in figure 25.6. The C-1000 normally operates with a cardioid directivity. Slip on the adapter and we lengthen the path between the front and rear of the diaphragm. Think of a microphone diaphragm in tube. In centre of tube we get a figure of 8 response. At one end of tube we get a cardioid response. Moving diaphragm towards centre of tube by adding short length of tube in front of the diaphragm will increase the figure of 8 part of the response. So we go from cardioid to hyper-cardioid. This would be useful if for example you were recording over-heads on a drum kit, and you wanted to pick up a bit more of the room.

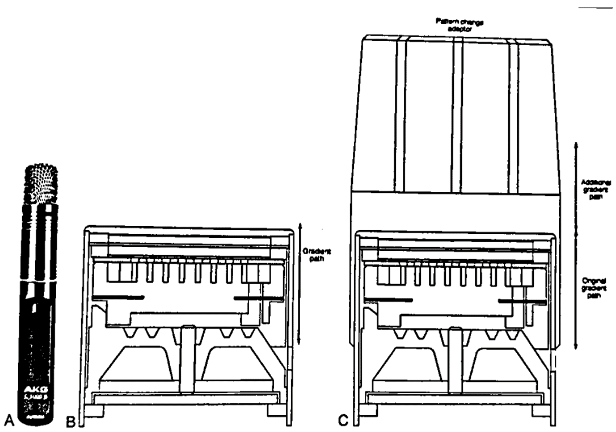


Figure 25.6: Extension capsule for AKG C-1000.

25.2.3 Sensitivity

Now lets think about the voltage output of a microphone. Why? Because we are interested in a) its sensitivity and b) its frequency response. In particular lets consider the capacitor microphone.

We have already seen that the output voltage of a capacitor microphone is proportional to the diaphragm spacing, bias voltage and the dynamic diaphragm displacement,

$$V = \frac{V_0}{X_0} x. \tag{25.39}$$

Note that the diaphragm displacement is related to its velocity by differentiation,

$$u = \frac{dx}{dt} \rightarrow x = \frac{u}{j\omega} \tag{25.40}$$

and that velocity and volume velocity are related by a factor of surface area,

$$x = \frac{U}{j\omega S}. \tag{25.41}$$

Substituting the above intro equation 25.39, and dividing both sides by pressure yields,

$$\frac{V}{p_f} = \frac{V_0}{j\omega S X_0} \frac{U}{p_f}. \tag{25.42}$$

Now we have just gone through a lot of effort to derive equation 25.29 for the acoustic transfer function, so lets substitute it in. Considering the cardioid response in particular, we have,

$$\frac{V}{p_f} = \frac{V_0}{cSX_0} \frac{d(1+\cos\theta)}{Z_1\,(1+jkd) + R_{hole}}. \tag{25.43}$$

Now lets substitute in for the mechanical impedance of the diaphragm (i.e. Z_1),

$$\frac{V}{p_f} = \frac{V_0}{cSX_0} \frac{d(1+\cos\theta)}{\left(j\omega M + R + \frac{1}{j\omega C}\right)(1+jkd) + R_{hole}}. \tag{25.44}$$

Assuming low frequencies $kd << 1$, the above simplifies to,

$$\frac{V}{p_f} = \frac{V_0}{cSX_0} \frac{d(1+\cos\theta)}{\left(j\omega M + R + \frac{1}{j\omega C}\right) + R_{hole}}. \tag{25.45}$$

Equation 25.45 represents the low frequency sensitivity of our capacitor microphone. Now what do we have to do to get a flat frequency response out of this microphone? The mass term gives us a -6dB slope, and the stiffness a $+6$dB slope. To get a flat response we have to use damping control. Basically, damp the hell out of it. So the mechanical and rear vent damping are the key parameters.

26 Dual Diaphragm Design

We have looked at utilising a variable rear opening to control the directivity of a microphone. When fully closed we got an omni directional response, and when fully open we got a figure of 8 response. Somewhere in-between we got a cardioid. It turns out you can achieve the same sort of controllability another way. Instead of using a variable opening, you can use a second diaphragm. By varying the voltage contribution of this second diaphragm you can in fact achieve the same sort of controllability as with the variable opening.

Here is the general idea. By having two diaphragms we will get expressions for the voltage response for each. These will be functions of their respective bias voltages. The output of the microphone will be the combined voltage output of the two diaphragms. By controlling the bias voltages we can control the magnitude of each contribution. If the bias voltages are the same, we should get an omni response. If the bias of the rear is that same but inverted, we should get a figure of 8. If the bias of the rear is 0 we should get a cardioid response. This makes sense right? By setting the second diaphragm's bias to 0, it should simply act as a resonant mass-spring system, just like the rear vent and cavity in our acoustic filtering network. Now lets go through the maths and see if this all works...

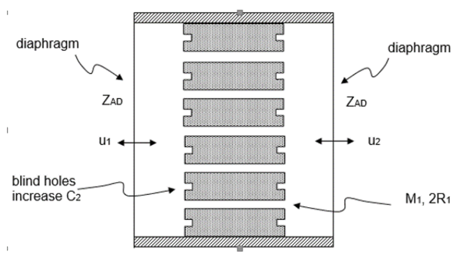


Figure 26.1: Dual diaphragm microphone design.

Shown in figure 26.1 is a schematic of a dual diaphragm microphone. The two diaphragms each have their own velocities and are separated by a pair of cavities with a series of small openings connecting them. These small openings act as an acoustic mass with some resistance. The cavities clearly act as compliances. The equivalent circuit for this microphone is in figure 26.2. It is driven by two pressure sources. These pressure sources drive the diaphragm impedances Z_{AD_1} and Z_{AD_2}. The cavity impedance is then split and separated by the acoustic mass/resistance of the small openings. Unfortunately, this circuit is quite awkward

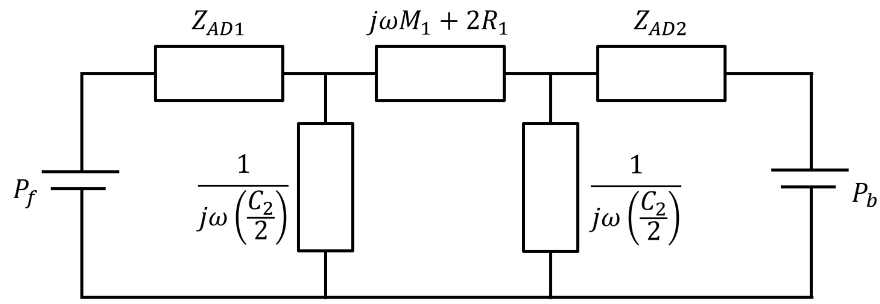
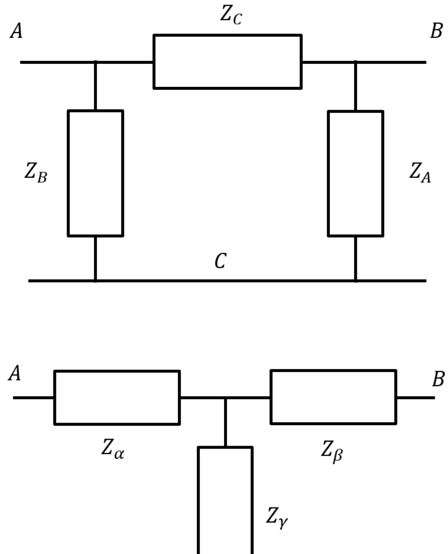


Figure 26.2: Equivalent circuit for dual diaphragm microphone design.

to work with. We want to try and simplify the central compliance/mass section. We can do this using what is called a Delta-Star transformation.



Circuit schematics like those in the top of figure 26.3 are often encountered in electrical networks. They are however tricky to deal with. As a means of simplifying electrical networks we can use the Delta-Star transformation. This lets us convert the tricky top circuit, into the simple bottom circuit.

Listed below are a series of impedance relations between the two circuits:

- Between terminals A and B,

$$\frac{Z_C(Z_A+Z_B)}{Z_A+Z_B+Z_C} = Z_\alpha + Z_\beta \tag{26.1}$$

- Between terminals A and C,

$$\frac{Z_B(Z_A+Z_C)}{Z_A+Z_B+Z_C} = Z_\alpha + Z_\gamma \tag{26.2}$$

- Between terminals B and C,

$$\frac{Z_A(Z_B+Z_C)}{Z_A+Z_B+Z_C} = Z_\beta + Z_\gamma \tag{26.3}$$

We will need to use all three moving forward, but for the sake of time we will only focus in detail on the top equation.

Equation 26.1 represents the total impedance between terminals A and B in both circuits. The left hand side corresponds to the upper (delta) circuit, and the right to the lower (star) circuit. Consider first the delta circuit. From inspection we can see that the impedance between A and B is that of Z_C in parallel with the remaining two impedances Z_B and Z_A. These remaining two impedances are in series with one another and so their combined impedance is just their sum. All together then, using the product over sum rule, we have the impedance of Z_C multiplied by $Z_B + Z_A$, all divided by their sum, $Z_C + Z_B + Z_A$. This gives us the left hand side of equation 26.1. From the star circuit we can see that the impedance between A and B of the transformed circuit is just the series impedance $Z_\alpha + Z_\beta$.

For the two circuit schematics to be equivalent they must have the same impedance across all terminal pairs. And so we have a total of three equations that need to be satisfied. To transform our equivalent circuit to the simpler 'star' form we need to determine the three impedances Z_α, Z_β, and Z_γ, i.e. solve the simultaneous equations. This isn't particularly fun, so ill give you the answers.

$$Z_\alpha = \frac{Z_B Z_C}{Z_A+Z_B+Z_C} \tag{26.4}$$

$$Z_\beta = \frac{Z_A Z_C}{Z_A+Z_B+Z_C} \tag{26.5}$$

$$Z_\gamma = \frac{Z_A Z_B}{Z_A+Z_B+Z_C} \tag{26.6}$$

These are the conversions we need to transform our circuit to the simpler form. Now lets apply the above to our equivalent circuit.

To make life a little easier for ourselves, let us assume that the mass effect of the small openings is negligible. Substituting in for our three impedances we get,

$$Z_\alpha = \frac{\frac{2R_1}{j\omega\frac{C}{2}}}{\frac{1}{j\omega\frac{C}{2}} + \frac{1}{j\omega\frac{C}{2}} + 2R_1} = \frac{\frac{4R_1}{j\omega C}}{\frac{4}{j\omega C} + 2R_1} = \frac{R_1}{1 + j\omega\frac{CR_1}{2}} \tag{26.7}$$

$$Z_\beta = \frac{R_1}{1 + j\omega \frac{CR_1}{2}} \tag{26.8}$$

$$Z_\gamma = \frac{\frac{4}{-\omega^2 C_2^2}}{\frac{1}{j\omega \frac{C}{2}} + \frac{1}{j\omega \frac{C}{2}} + 2R_1} = \frac{\frac{4}{-\omega^2 C_2^2}}{\frac{4}{j\omega C} + 2R_1} = \frac{\frac{1}{j\omega C_2}}{1 + j\omega \frac{CR_1}{2}} \tag{26.9}$$

After applying the delta-star transformation above, the equivalent circuit now takes the form of figure 26.4.

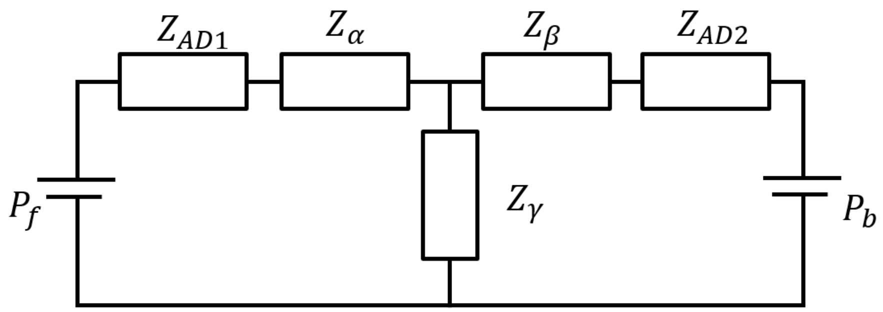


Figure 26.4: Equivalent circuit for dual diaphragm microphone design with delta-star transformation.

By denoting,

$$Z_1 = Z_{AD_1} + Z_\alpha \tag{26.10}$$

$$Z_2 = Z_\gamma \tag{26.11}$$

$$Z_3 = Z_{AD_2} + Z_\beta \tag{26.12}$$

we arrive at an equivalent circuit identical in form to our single diaphragm condenser with an acoustic filtering network, as in figure 25.3, except now the rear volume velocity corresponds to the second diaphragm.

Recalling the acoustic transfer function from equation 25.30,

$$\frac{U}{P_f} = \frac{Z_3 + jkdZ_2 \cos\theta}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}. \tag{26.13}$$

If the diaphragms is made sufficiently low, their contribution can be neglected entirely, and we have that,

$$Z_1 = \frac{R_1}{1 + j\omega \frac{C_2 R_1}{2}} \tag{26.14}$$

$$Z_2 = \frac{\frac{1}{j\omega C_2}}{1 + j\omega \frac{C_2 R_1}{2}} \tag{26.15}$$

$$Z_3 = \frac{R_1}{1 + j\omega \frac{C_2 R_1}{2}}. \tag{26.16}$$

Now to achieve a cardioid response for the front diaphragm, we need $Z_3 = jkdZ_2$,

$$Z_3 = jkdZ_2 \rightarrow \frac{R_1}{1 + j\omega \frac{C_2 R_1}{2}} = \frac{\frac{jkd}{j\omega C_2}}{1 + j\omega \frac{C_2 R_1}{2}} \rightarrow R_1 = \frac{jkd}{j\omega C_2} = \frac{d}{cC_2}. \tag{26.17}$$

Using this cardioid design lets consider the low frequency transfer function where $kd << 1$. At low frequencies each impedance term as can approximated as so,

$$Z_1 = Z_3 \frac{\frac{d}{cC_2}}{1 + j\omega \frac{C_2 \frac{d}{cC_2}}{2}} = \frac{\frac{d}{cC_2}}{1 + \frac{jkd}{2}} \xrightarrow{kd<<1} \frac{d}{cC_2} = R_1 \tag{26.18}$$

$$Z_2 = \frac{\frac{1}{j\omega C_2}}{1 + j\omega \frac{C_2 \frac{d}{cC_2}}{2}} = \frac{\frac{1}{j\omega C_2}}{1 + \frac{jkd}{2}} \xrightarrow{kd<<1} \frac{1}{j\omega C_2}. \tag{26.19}$$

Now we can put together our low frequency equivalent circuit for a dual diaphragm capacitor microphone. This is much simpler than where we started. Note that we have designed this microphone such that the contribution from the rear diaphragm yields a cardioid response on the front diaphragm. We have not yet considered the effect of taking the combined output of both diaphragms, nor varying their bias voltages. This is what will give us a variable directivity.

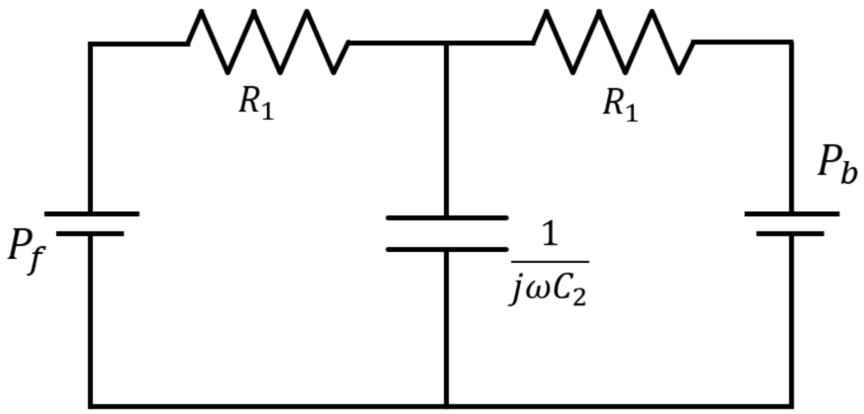


Figure 26.5: Low frequency equivalent circuit for dual diaphragm microphone design.

Remember that the dynamic voltage output from a capacitor microphone can be given by,

$$\frac{V}{p_f} = \frac{V_0}{j\omega S X_0} \frac{U}{p_f}. \tag{26.20}$$

Substituting in equation 25.30, along with the low frequency impedance terms in equations 26.18 and 26.19, yields,

$$\frac{V}{p_f} = \frac{V_0}{j\omega S X_0} \frac{\frac{d}{cC_2} + \frac{jkd}{j\omega C_2}\cos\theta}{\frac{\frac{d}{cC_2}}{j\omega C_2} + \left(\frac{d}{cC_2}\right)^2 + \frac{\frac{d}{cC_2}}{j\omega C_2}} = \frac{V_0}{j\omega S X_0} \frac{C_2}{\frac{d}{c} + \frac{2}{j\omega}}(1+\cos\theta). \tag{26.21}$$

After some simply rearranging,

$$\frac{V}{p_f} = \frac{V_0}{j\omega S X_0} \frac{C_2}{\frac{d}{c} + \frac{2}{j\omega}}(1+\cos\theta) = \frac{V_0}{j\omega S X_0} \frac{C_2}{jkd+2}(1+\cos\theta) \tag{26.22}$$

we have the pressure to voltage transfer function, i.e. the microphone sensitivity. Remember we are considering low frequencies, so we can neglect the jkd in the denominator. Our low frequency sensitivity is then

$$\frac{V}{p_f} = \frac{V_0}{j\omega S X_0} \frac{C_2}{2}(1+\cos\theta). \tag{26.23}$$

Note that it is proportional to the bias voltage. Also note that this bias voltage is that applied to the front diaphragm, $V_0 = V_f$. The rear diaphragm will have its own bias voltage.

We have now derived a sensitivity relation that is valid for both the front and rear diaphragms. To make things a bit clearer lets also define a new constant term $K = C_2/2X_0S$.

It is important to remember however that the incident angle for the front and rear diaphragms will not be the same. The rear diaphragm will be offset by a factor of π, as illustrated in figure 26.6.

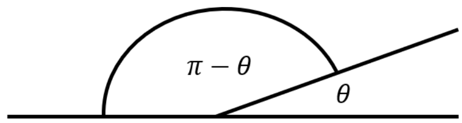


Figure 26.6: Offset angle for rear pressure.

Also remember that the front and rear pressures will not be equal, they will differ by a term proportional to the time of flight/angle $jkd\cos\theta$. However, at low frequencies, this term becomes negligible and we can say that $P_f \approx P_b$.

The front diaphragm sensitivity is,

$$\frac{v_f}{p_f} = V_f K(1 + \cos\theta) \tag{26.24}$$

and the rear diaphragm sensitivity is,

$$\frac{v_b}{p_b} = V_b K(1 + \cos(\pi - \theta)) = V_b K(1 - \cos\theta). \tag{26.25}$$

The total output, or the total sensitivity, is the sensitivity corresponding to the summed output voltages. Adding together our two sensitivities and regrouping terms accordingly we arrive at,

$$\frac{v_T}{p_f} = \frac{v_f + v_b}{p_f} = K\left[(V_f + V_b) + (V_f - V_b)\cos\theta\right]. \tag{26.26}$$

Now that we have our total output, lets look at what happens as we alter the bias voltage across the rear diaphragm.

Suppose the two bias voltages are equal, $V_b = V_f$. In this case the second term equals 0 and we get that,

$$\frac{v_T}{p_f} = 2KV_f. \tag{26.27}$$

This is an omni directional response. It has no dependence on angle. Now suppose the two bias voltages are equal and opposite, $V_b = -V_f$. In this case the first term is 0 and we get,

$$\frac{v_T}{p_f} = 2KV_f\cos\theta. \tag{26.28}$$

This is a figure of 8 response. It has a directivity of $\cos\theta$. Now what if the rear bias voltage is equal to 0? Well this is how we designed our microphone to start with; both terms contribute and we get,

$$\frac{v_T}{p_f} = 2KV_f(1 + \cos\theta). \tag{26.29}$$

This is a cardioid response. It has a directivity of $1 + \cos\theta$.

So by simply changing the voltage across the rear diaphragm we can control the directivity. Neat trick ay? Note that you can also achieve other first order directivity patterns by setting the rear bias voltage to something a little different.

27 Boundary Effects

Suppose we place a microphone some distance d above a surface. What will we record? We will have direct sound component, but also not so helpfully a reflection arriving from the surface below.

In general, reflections are bad news. Their difference in path length compared to the direct sound can cause comb filtering, which is definitely bad news.

Hard reflective boundaries tend to act a lot like mirrors, producing specular reflections (i.e. reflections that leave at the same angle they arrived). To be honest, most surfaces (apart from fluffy stuff) act as if they are mirrors (perhaps dirty mirrors?).

There is another interesting way of thinking about reflections. Boundaries act like mirrors right? Instead of thinking of a microphone and a hard reflective surface, we can think of two microphones, one being the mirror image of the other (the hard surface being the mirror). This second microphone is positioned the same distance form the mirror as its counterpart. Together the combined signal from both microphones appears identical to that of a single microphone with a reflective boundary.

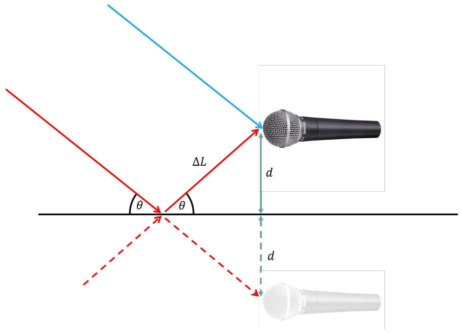


Figure 27.1: Reflection of a hard surface.

It is clear from figure 27.1 that the reflected wave has to travel an extra distance ΔL. Because ΔL is the same for the virtual mic and the real one (same triangles) it is if it just travels through the wall to another receiver.

So if we can think of the total response as being the sum from both microphones, can we think of these two microphone as a mini array? Sure.

Lets assume plane waves, i.e. the sound originates far way, in this case we have,

$$\Delta L = 2.d \sin\theta \tag{27.1}$$

This makes sense right? If sound is coming in from the side, $\theta = 0$, then the wave hits both microphones at same time $(\sin 0 = 0)$. If sound is coming in from front, $\theta = 90$, then the wave passes one microphone, then there is a path difference of $2d$ before the next. This works at any angle.

From this and the wave number we can get the phase difference of the reflected wave,

$$\phi = k2d \sin\theta. \tag{27.2}$$

When you have a time delay between signal and a repetition of itself, at a particular set of frequencies we get $180°$ phase delay, which causes massive cancellations. At another set of frequencies they are in phase, giving us constructive interference! What's the end result? An effect we call comb filtering, i.e. a series of peaks and troughs in the frequency response of the microphone.

You can often hear this effect on news programs where the reporter is sat at desk (hard reflective surface!) You can actually get special tables (holy tables) with lots of holes in to prevent reflections. Another good example is if a singer is using a lyric sheet. The resulting reflections can cause colouration. In this case the worst place to have a notch is around 1-2 kHz, i.e. a distance of the order of 5-15 cm.

With comb filtering you get a null when the delay is half a wavelength. Then you get a peak when the path delay is the same as the wavelength. This pattern repeats as you increase frequency. You get nulls at odd integer multiples of half a wavelength, and peaks at even integer multiples of a wavelength, as in figure 27.2.

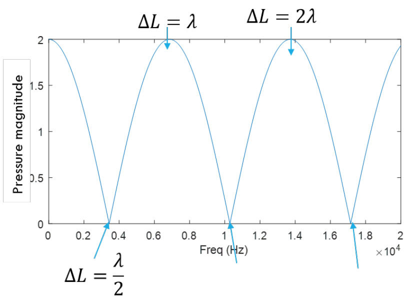


Figure 27.2: Reflection of a hard surface.

So boundaries are clearly a problem. How can we deal with them? There are two solutions to the problem: 1) Stay away from boundaries – prevent comb filtering and 2) get very very close to the boundary – push the first null beyond the frequency range of interest.

Why isn't comb filtering a problem in the middle of the room? You are still getting lots of reflections? If delays are long enough, you get lots and lots of nulls, i.e. a very 'fast' oscillation in frequency response, so many in fact that we have multiple within each of our critical bands so in the end it all averages out.

Our second remedy is to place the microphone very close to the boundary. In this case the first notch is very high, beyond our frequency range of interest. Remember, the first notch occurs when wavelength is 2 times the distance. By mounting microphone pretty much on boundary you don't get any interference due to boundary interactions.

There is a another advantage gained by placing a microphone against a boundary? The air pressure at the boundary is doubled. So you get twice as much output. This gives us a 6dB boost. What happens if we add another boundary (e.g. put the microphone in a corner between two walls)? We get 4 times the pressure! That's a 12dB boost. Now what about a corner between three walls? 18dB.

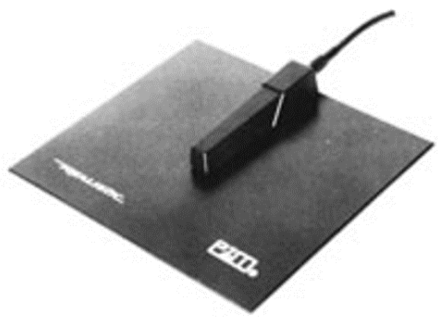


Figure 27.3: Boundary microphone.

Thing to remember is that this effect is predominantly a low frequency one, i.e. when the microphone behaves omni-directionally. Also, if the boundary is small then not all the wave gets reflected at low frequencies and the pressure does not add up. The boundary layer microphone acts like a low frequency shelving filter. At high frequencies how ill it change? The polar pattern will get narrower, so we expect it to be more directional due to finite size of diaphragm.